Group Theory

Problem Set 1 October 12, 2001

Note: Problems marked with an asterisk are for Rapid Feedback; problems marked with a double asterisk are optional.

1. Show that the wave equation for the propagation of an impulse at the speed of light c,

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\,,$$

is covariant under the Lorentz transformation

$$x' = \gamma(x - vt), \quad y' = y, \quad z' = z, \quad t' = \gamma\left(t - \frac{v}{c^2}x\right),$$

where $\gamma = (1 - v^2/c^2)^{-1/2}$.

2.* The Schrödinger equation for a free particle of mass m is

$$i\hbar\frac{\partial \varphi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2 \varphi}{\partial x^2}\,.$$

Show that this equation is invariant to the global change of phase of the wavefunction:

$$\varphi \to \varphi' = \mathrm{e}^{i\alpha}\varphi\,,$$

where α is any real number. This is an example of an **internal** symmetry transformation, since it does not involve the space-time coordinates.

According to Noether's theorem, this symmetry implies the existence of a conservation law. Show that the quantity $\int_{-\infty}^{\infty} |\varphi(x,t)|^2\, dx$ is independent of time for solutions of the free-particle Schrödinger equation.

3.* Consider the following sets of elements and composition laws. Determine whether they are groups and, if not, identify which group property is violated.

(a) The rational numbers, excluding zero, under multiplication.

(b) The non-negative integers under addition.

(c) The even integers under addition.

(d) The nth roots of unity, i.e., $\mathrm{e}^{2\pi m i/n}$, for $m = 0, 1, \ldots, n-1$, under multiplication.

(e) The set of integers under ordinary *subtraction*.

4.** The general form of the Liouville equation is

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[p(x)\frac{\mathrm{d}y}{\mathrm{d}x}\right] + \big[q(x) + \lambda r(x)\big]y = 0$$

where p, q and r are real-valued functions of x with p and r taking only positive values. The quantity λ is called the eigenvalue and the function y, called the eigenfunction, is assumed to be defined over an interval $[a, b]$. We take the boundary conditions to be

$$y(a) = y(b) = 0$$

but the result derived below is also valid for more general boundary conditions. Notice that the Liouville equations contains the one-dimensional Schrödinger equation as a special case.

Let $u(x;\lambda)$ and $v(x;\lambda)$ be the fundamental solutions of the Liouville equation, i.e. u and v are two linearly-independent solutions in terms of which all other solutions may be expressed (for a given value λ). Then there are constants A and B which allow any solution y to be expressed as a linear combination of this fundamental set:

$$y(x;\lambda) = Au(x;\lambda) + Bv(x;\lambda)$$

These constants are determined by requiring $y(x;\lambda)$ to satisfy the boundary conditions:

$$y(a;\lambda) = Au(a;\lambda) + Bv(a;\lambda) = 0$$

$$y(b;\lambda) = Au(b;\lambda) + Bv(b;\lambda) = 0$$

Use this to show that the solution $y(x;\lambda)$ is unique, i.e., that there is one and only one solution corresponding to an eigenvalue of the Liouville equation.

Group Theory

Solutions to Problem Set 1 October 6, 2000

1. To express the the wave equation for the propagation of an impulse at the speed of light c,

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\,, \tag{1}$$

under the Lorentz transformation

$$x' = \gamma(x - vt), \quad y' = y, \quad z' = z, \quad t' = \gamma\Big(t - \frac{v}{c^2}x\Big)\,, \tag{2}$$

where $\gamma = (1 - v^2/c^2)^{-1/2}$, we need to obtain expressions for the second derivatives in the primed variables. With $u'(x', y', z') = u(x, y, t)$, we have

$$\frac{\partial u}{\partial x} = \frac{\partial u'}{\partial x'}\frac{\partial x'}{\partial x} + \frac{\partial u'}{\partial t'}\frac{\partial t'}{\partial x} = \gamma\frac{\partial u'}{\partial x'} - \gamma\frac{v}{c^2}\frac{\partial u'}{\partial t'}\,, \tag{3}$$

$$\frac{\partial^2 u}{\partial x^2} = \gamma^2\frac{\partial^2 u'}{\partial x'^2} + \gamma^2\frac{v^2}{c^4}\frac{\partial^2 u'}{\partial t'^2}\,, \tag{4}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u'}{\partial y'^2}\,, \tag{5}$$

$$\frac{\partial^2 u}{\partial z^2} = \frac{\partial^2 u'}{\partial z'^2}\,, \tag{6}$$

$$\frac{\partial u}{\partial t} = \frac{\partial u'}{\partial t'}\frac{\partial t'}{\partial t} + \frac{\partial u'}{\partial x'}\frac{\partial x'}{\partial t} = \gamma\frac{\partial u'}{\partial t'} - \gamma v\frac{\partial u'}{\partial x'}\,, \tag{7}$$

$$\frac{\partial^2 u}{\partial t^2} = \gamma^2\frac{\partial^2 u'}{\partial t'^2} + \gamma^2 v^2\frac{\partial^2 u'}{\partial x'^2}\,. \tag{8}$$

Substituting these expressions into the wave equation yields

$$\frac{\gamma^2}{c^2}\frac{\partial^2 u'}{\partial t'^2} + \frac{\gamma^2 v^2}{c^2}\frac{\partial^2 u'}{\partial x'^2} = \gamma^2\frac{\partial^2 u'}{\partial x'^2} + \gamma^2\frac{v^2}{c^4}\frac{\partial^2 u'}{\partial t'^2} + \frac{\partial^2 u'}{\partial y'^2} + \frac{\partial^2 u'}{\partial z'^2}\,, \tag{9}$$

which, upon rearrangement, becomes

$$\frac{\gamma^2}{c^2}\left(1-\frac{v^2}{c^2}\right)\frac{\partial^2 u'}{\partial t'^2} = \gamma^2\left(1-\frac{v^2}{c^2}\right)\frac{\partial^2 u'}{\partial x'^2} + \frac{\partial^2 u'}{\partial y'^2} + \frac{\partial^2 u'}{\partial z'^2}\,. \tag{10}$$

Invoking the definition of γ, we obtain

$$\frac{1}{c^2}\frac{\partial^2 u'}{\partial t'^2} = \frac{\partial^2 u'}{\partial x'^2} + \frac{\partial^2 u'}{\partial y'^2} + \frac{\partial^2 u'}{\partial z'^2}\,, \tag{11}$$

which confirms the covariance of the wave equation under the Lorentz transformation.

2. We begin with the Schrödinger equation for a free particle of mass m:

$$i\hbar\frac{\partial\varphi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\varphi}{\partial x^2}\,. \tag{12}$$

Performing the transformation

$$\varphi \to \varphi' = \mathrm{e}^{i\alpha}\varphi\,, \tag{13}$$

where α is any real number, we find

$$i\hbar\mathrm{e}^{i\alpha}\frac{\partial\varphi'}{\partial t} = -\frac{\hbar^2}{2m}\mathrm{e}^{i\alpha}\frac{\partial^2\varphi'}{\partial x^2}\,, \tag{14}$$

or, upon cancelling the common factor $\mathrm{e}^{i\alpha}$,

$$i\hbar\frac{\partial\varphi'}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\varphi'}{\partial x^2}\,, \tag{15}$$

which establishes the covariance of the Schrödinger equation under this transformation.

We can derive the corresponding quantity multiplying Eq. (12) by the complex conjugate φ^* and subtracting the product of φ and the complex conjugate of Eq. (12):

$$i\hbar\left(\varphi^*\frac{\partial\varphi}{\partial t} + \varphi\frac{\partial\varphi^*}{\partial t}\right) = -\frac{\hbar^2}{2m}\left(\varphi^*\frac{\partial^2\varphi}{\partial x^2} - \varphi\frac{\partial^2\varphi^*}{\partial x^2}\right). \tag{16}$$

The left-hand side of this equation can be written as

$$i\hbar\left(\varphi^*\frac{\partial\varphi}{\partial t}+\varphi\frac{\partial\varphi^*}{\partial t}\right)=i\hbar\frac{\partial}{\partial t}\left(\varphi\varphi^*\right)=i\hbar\frac{\partial}{\partial t}|\varphi|^2\,, \tag{17}$$

and the right-hand side can be written as

$$-\frac{\hbar^2}{2m}\left(\varphi^*\frac{\partial^2\varphi}{\partial x^2}-\varphi\frac{\partial^2\varphi^*}{\partial x^2}\right)=-\frac{\hbar^2}{2m}\frac{\partial}{\partial x}\left(\varphi^*\frac{\partial\varphi}{\partial x}-\varphi\frac{\partial\varphi^*}{\partial x}\right). \tag{18}$$

We now consider the solution to Eq. (12) corresponding to a **wave packet**, whereby by φ and its derivatives vanish as $x \to \pm\infty$. Then, integrating Eq. (16) over the real line, and using Eqns. (17) and (18), we obtain

$$\begin{aligned} i\hbar\frac{\partial}{\partial t}\int_{-\infty}^{\infty}|\varphi(x,t)|^2\,dx &= -\frac{\hbar^2}{2m}\int_{-\infty}^{\infty}\left[\frac{\partial}{\partial x}\left(\varphi^*\frac{\partial\varphi}{\partial x}-\varphi\frac{\partial\varphi^*}{\partial x}\right)\right]dx \\ &= -\frac{\hbar^2}{2m}\left(\varphi^*\frac{\partial\varphi}{\partial x}-\varphi\frac{\partial\varphi^*}{\partial x}\right)\Bigg|_{-\infty}^{\infty} \\ &= 0\,. \end{aligned} \tag{19}$$

Thus, the quantity $\int_{-\infty}^{\infty}|\varphi(x,t)|^2\,dx$ is independent of time for solutions of the free-particle Schrödinger equation which correspond to wave packets.

3. (a) The multiplication of two rational numbers, m/n and p/q, where m, n, p and q are integers, yields another rational number, mp/nq, so closure is obeyed. The unit is 1, multiplication is an associative operation, and the inverse of m/n is n/m, which is a rational number. The set excludes zero, so the problem of finding the inverse of zero does not arise. Hence, the rational numbers, excluding zero, under multiplication form a group.

   (b) The sum of two non-negative integers is a non-negative integer, thus ensuring closure, addition is associative, the unit is zero, but the inverse under addition of a negative integer n is $-n$, which is a *negative* integer and, therefore, excluded from the set. Hence, the non-negative integers under addition do not form a group.

(c) The sum of two even integers $2m$ and $2n$, where m and n are any two integers, is $2(m+n)$, which is an even integer, so closure is obeyed. Addition is associative, the unit is zero, which is an even integer, and the inverse of $2n$ is $-2n$, which is also an even integer. Hence, the even integers under addition form a group.

(d) The multiplication of two elements amounts to the addition of the integers $0, 1, \ldots, n-1$, modulo n, i.e., the addition of any two elements and, if the sum lies out side of this range, subtract n to bring it into the range. Thus, the multiplication of two nth roots of unity is again an nth root of unity, multiplication is associative, and the unit is 1. The inverse of $\mathrm{e}^{2\pi mi/n}$ is therefore

$$\mathrm{e}^{-2\pi mi/n} = \mathrm{e}^{2\pi i(n-m)/n} \tag{20}$$

Thus, the nth roots of unity form a group for any value of n.

(e) For the set of integers under ordinary subtraction, the difference between two integers n and m is another integer p, $n-m=p$, the identity is 0, since, $n-0=0$, and every integer is its own inverse, since $n-n=0$. However, subtraction is not associative because

$$(n-m)-p \neq n-(m-p) = n-m+p$$

Hence, the integers under ordinary subtraction do not form a group.

4. Let $u(x;\lambda)$ and $v(x;\lambda)$ be the fundamental solutions of the Liouville equation with the boundary conditions $y(a)=y(b)=0$. Then there are constants A and B which allow any solution y to be expressed as a linear combination of this fundamental set:

$$y(x;\lambda) = Au(x;\lambda) + Bv(x;\lambda) \tag{21}$$

These constants are determined by requiring $y(x;\lambda)$ to satisfy the boundary conditions given above. Applying these boundary conditions to the expression in (21) leads to the following equations:

$$\begin{aligned} y(a;\lambda) &= Au(a;\lambda) + Bv(a;\lambda) = 0 \\ y(b;\lambda) &= Au(b;\lambda) + Bv(b;\lambda) = 0 \end{aligned} \tag{22}$$

Equations (22) are two simultaneous equations for the unknown quantities A and B. To determine the conditions which guarantee that this system of equations has a nontrivial solution, we write these equations in matrix form:

$$\begin{pmatrix} u(a;\lambda) & v(a;\lambda) \\ u(b;\lambda) & v(b;\lambda) \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{23}$$

Thus, we see that Equations (22) can be solved for nonzero values of A and B only if the determinant of the matrix of coefficients in (23) vanishes. Otherwise, the only solution is $A = B = 0$, which yields the trivial solution $y = 0$. The condition for a nontrivial solution of (22) is, therefore, given by

$$\begin{vmatrix} u(a;\lambda) & v(a;\lambda) \\ u(b;\lambda) & v(b;\lambda) \end{vmatrix} = u(a;\lambda)v(b;\lambda) - u(b;\lambda)v(a;\lambda) = 0 \tag{24}$$

This guarantees that the solution for A and B is unique. Hence, the eigenvalues are non-degenerate.

Group Theory

Problem Set 2 October 16, 2001

Note: Problems marked with an asterisk are for Rapid Feedback.

1.* Show that, by requiring the existence of an identity in a group G, it is sufficient to require only a *left* identity, $ea = a$, or only a *right* identity $ae = a$, for every element a in G, since these two quantities *must* be equal.

2.* Similarly, show that it is sufficient to require only a *left* inverse, $a^{-1}a = e$, or only a *right* inverse $aa^{-1} = e$, for every element a in G, since these two quantities must also be equal.

3. Show that for any group G, $(ab)^{-1} = b^{-1}a^{-1}$.

4.* For the elements $g_1, g_2, \ldots, g_n$ of a group, determine the inverse of the n-fold product $g_1 g_2 \cdots g_n$.

5.* Show that a group is Abelian if and only if $(ab)^{-1} = a^{-1}b^{-1}$. You need to show that this condition is both necessary and sufficient for the group to be Abelian.

6. By explicit construction of multiplication tables, show that there are two distinct structures for groups of order 4. Are either of these groups Abelian?

7.* Consider the group of order 3 discussed in Section 2.4. Suppose we regard the rows of the multiplication table as individual permutations of the elements $\{e, a, b\}$ of this group. We label the permutations π_g by the group element corresponding to that row:

$$\pi_e = \begin{pmatrix} e & a & b \\ e & a & b \end{pmatrix}, \qquad \pi_a = \begin{pmatrix} e & a & b \\ a & b & e \end{pmatrix}, \qquad \pi_b = \begin{pmatrix} e & a & b \\ b & e & a \end{pmatrix}$$

(a) Show that, under the composition law for permutations discussed in Section 2.3, the multiplication table of the 3-element group is preserved by this association, e.g., $\pi_a \pi_b = \pi_e$.

(b) Show that for every element g in $\{e, a, b\}$,

$$\pi_g = \begin{pmatrix} e & a & b \\ g & ga & gb \end{pmatrix}$$

Hence, show that the π_g have the same multiplication table as the 3-element group.

(c) Determine the relationship between this group and S_3. This is an example of Cayley's theorem.

(d) To which of the operations on an equilateral triangle in Fig. 2.1 do these group elements correspond?

Group Theory

Solutions to Problem Set 2 October 26, 2001

1. Suppose that e is the right identity of a group G,

$$ge = g \tag{1}$$

for all g in G, and that e' is the left identity,

$$e'g = g \tag{2}$$

for all g in G. The choice $g = e'$ in the first of these equations and $g = e$ in the second, yields

$$e'e = e' \tag{3}$$

and

$$e'e = e \tag{4}$$

respectively. These equations imply that

$$e' = e \tag{5}$$

so the left and right identities are equal. Hence, we need specify only the left *or* right identity in a group in the knowledge that this is *the* identity of the group.

2. Suppose that a is the right inverse of any element g in a group G,

$$ga = e \tag{6}$$

and a' is the left inverse of g,

$$a'g = e \tag{7}$$

Multiplying the first of these equations from the left by a' and invoking the second equation yields

$$a' = a'(ga) = (a'g)a = a \tag{8}$$

so the left and right inverses of an element are equal. The same result could have been obtained by multiplying the second equation from the right by a and invoking the first equation.

3. To show that for any group G, $(ab)^{-1} = b^{-1}a^{-1}$, we begin with the properties of the inverse. We must have that

$$(ab)(ab)^{-1} = e$$

Left-multiplying both sides of this equation first by a^{-1} and then by b^{-1} yields

$$(ab)^{-1} = b^{-1}a^{-1}$$

4. For elements $g_1, g_2, \ldots, g_n$ of a group G, we require the inverse of the n-fold product $g_1 g_2 \cdots g_n$. We proceed as in Problem 2 using the definition of the inverse to write

$$(g_1 g_2 \cdots g_n)(g_1 g_2 \cdots g_n)^{-1} = e$$

We now follow the same procedure as in Problem 2 and left-multiply both sides of this equation in turn by $g_1^{-1}, g_2^{-1}, \ldots, g_n^{-1}$ to obtain

$$(g_1 g_2 \cdots g_n)^{-1} = g_n^{-1} \cdots g_2^{-1} g_1^{-1}$$

5. We must prove two statements here: that for an Abelian group G, $(ab)^{-1} = a^{-1}b^{-1}$, for all a and b in G, and that this equality implies that G is Abelian. If G is Abelian, then using the result

of Problem 3 and the commutativity of the composition law, we find

$$(ab)^{-1} = b^{-1}a^{-1} = a^{-1}b^{-1}$$

Now, suppose that there is a group G (which we must not *assume* is Abelian, such that

$$(ab)^{-1} = a^{-1}b^{-1}$$

for all a and b in G. We now right multiply both sides of this equation first by b and then by a to obtain

$$(ab)^{-1}ba = e$$

Then, left-multiplying both sides of this equation by (ab) yields

$$ba = ab$$

so G is Abelian. Hence, we have shown that G is an Abelian group if and only if, for elements a and b in G, $(ab)^{-1} = a^{-1}b^{-1}$.

6. To construct the multiplication table of a four-element group $\{e, a, b, c\}$ we proceed as in Section 2.4 of the course notes. The properties of the unit of the group enable us to make the following entries into the multiplication table:

	e	a	b	c
e	e	a	b	c
a	a			
b	b			
c	c			

We now consider the product aa. This cannot equal a, since that would imply that $a = e$, but it can equal any of the other elements, including the identity. However, this leads only to two *distinct* choices for the product, since the apparent difference between $aa = b$ and $aa = c$ disappears under the interchange of the

	e	a	b	c
e	e	a	b	c
a	a	e		
b	b			
c	c			

	e	a	b	c
e	e	a	b	c
a	a	b		
b	b			
c	c			

(1)

labelling of b and c. Thus, at this stage, we have two distinct structures for the multiplication table:

We now determine the remaining entries for these two groups. For the table on the left, we consider the product ab. From the Rearrangement Theorem, this cannot equal a or or e, nor can it equal b (since that would imply $a = e$). Therefore, $ab = c$, from which it follows that $ac = b$. According to the Rearrangement Theorem, the multiplication table becomes

	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c		
c	c	b		

(2)

For the remaining entries of this table, we observe that $b^2 = a$ and $b^2 = e$ are equally valid assignments. These leads to two multiplication tables:

	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	a	e
c	c	b	e	a

(3)

Note that these tables are distinct in that there is no relabelling of the elements which transforms one into the other.

We now return to the other multiplication table on the right in (1). The Rearrangement Theorem requires that the second row must be completed as follows:

	e	a	b	c
e	e	a	b	c
a	a	b	c	e
b	b	c		
c	c	e		

(4)

Again invoking the Rearrangement Theorem, we must have that this multiplication table can be completed only as:

	e	a	b	c
e	e	a	b	c
a	a	b	c	e
b	b	c	e	a
c	c	e	a	b

(5)

Notice that all of the multiplication tables in (3) and (5) are Abelian and that the table in (5) is cyclic, i.e., all of the group elements can be obtained by taking successive products of any non-unit element.

We now appear to have three distinct multiplication tables for groups of order 4: the two tables in (3) and the one in (5). However, if we reorder the elements in the second table in (3) to $\{e, b, a, c\}$ and reassemble the multiplication table (using the same products), we obtain

	e	b	a	c
e	e	b	a	c
c	b	a	c	e
b	a	c	e	b
a	c	e	b	a

(6)

which, under the relabelling $a \mapsto b$ and $b \mapsto a$, is identical to (5). Hence, there are only two distinct structures of groups with four elements:

	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

	e	a	b	c
e	e	a	b	c
a	a	b	c	e
b	b	c	e	a
c	c	e	a	b

(7)

7. (a) All of the products involving the identity are self-evident. The only products that must be calculated explicitly are a^2, ab, ba, and b^2. These are given by

$$\begin{aligned}
\pi_a\pi_a &= \begin{pmatrix} e & a & b \\ a & b & e \end{pmatrix}\begin{pmatrix} e & a & b \\ a & b & e \end{pmatrix} = \begin{pmatrix} e & a & b \\ b & e & a \end{pmatrix} = \pi_b \\
\pi_a\pi_b &= \begin{pmatrix} e & a & b \\ a & b & e \end{pmatrix}\begin{pmatrix} e & a & b \\ b & e & a \end{pmatrix} = \begin{pmatrix} e & a & b \\ e & a & b \end{pmatrix} = \pi_e \\
\pi_b\pi_a &= \begin{pmatrix} e & a & b \\ b & e & a \end{pmatrix}\begin{pmatrix} e & a & b \\ a & b & e \end{pmatrix} = \begin{pmatrix} e & a & b \\ e & a & b \end{pmatrix} = \pi_e \\
\pi_b\pi_b &= \begin{pmatrix} e & a & b \\ b & e & a \end{pmatrix}\begin{pmatrix} e & a & b \\ b & e & a \end{pmatrix} = \begin{pmatrix} e & a & b \\ a & b & e \end{pmatrix} = \pi_a
\end{aligned}$$

Thus, the association $g \to \pi_g$, for $g = e, a, b$ preserves the products of the three-element group.

(b) With the construction

$$\pi_g = \begin{pmatrix} e & a & b \\ g & ga & gb \end{pmatrix} \tag{9}$$

for $g = e, a, b$, we can use the rows of the multiplication table on p. 19 to obtain

$$\begin{aligned}
\pi_e &= \begin{pmatrix} e & a & b \\ ee & ea & eb \end{pmatrix} = \begin{pmatrix} e & a & b \\ e & a & b \end{pmatrix} \\
\pi_a &= \begin{pmatrix} e & a & b \\ ae & aa & ab \end{pmatrix} = \begin{pmatrix} e & a & b \\ a & b & e \end{pmatrix}
\end{aligned}$$

$$\pi_b = \begin{pmatrix} e & a & b \\ be & ba & bb \end{pmatrix} = \begin{pmatrix} e & a & b \\ b & e & a \end{pmatrix} \qquad (10)$$

Thus, the association $g \to \pi_g$ is one-to-one and preserves the products of the 3-element group. Hence, these groups are equivalent.

(c) The elements of the three-element group correspond to the **cyclic permutations** of S_3. In other words, given a reference order $\{a, b, c\}$, the cyclic permutations are $a \mapsto b$, $b \mapsto c$, and $c \mapsto a$, yielding $\{b, c, a\}$, and then $b \mapsto c$, $c \mapsto a$, and $a \mapsto b$, yielding $\{c, a, b\}$.

(d) These elements correspond to the **rotations** of an equilateral triangle, i.e., the elements $\{e, d, f\}$ in Fig. 2.1.

Group Theory

Problem Set 3 October 23, 2001

Note: Problems marked with an asterisk are for Rapid Feedback.

1.* List all of the subgroups of any group whose order is a prime number.

2.* Show that a group whose order is a prime number is necessarily cyclic, i.e., all of the elements can be generated from the powers of any non-unit element.

3. Suppose that, for a group G, $|G| = pq$, where p and q are both prime. Show that every proper subgroup of G is cyclic.

4.* Let g be an element of a finite group G. Show that $g^{|G|} = e$.

5. In a quotient group G/H, which set *always* corresponds to the unit "element"?

6. Show that, for an Abelian group, every element is in a class by itself.

7. Show that every subgroup with index 2 is self-conjugate.

Hint: The conjugating element is either in the subgroup or not. Consider the two cases separately.

8.* Consider the following cyclic group of order 4, $G = \{a, a^2, a^3, a^4 = e\}$ (cf. Problem 6, Problem Set 2). Show, by direct multiplication or otherwise, that the subgroup $H = \{e, a^2\}$ is self-conjugate and identify the elements in the factor group G/H.

9.* Suppose that there is an isomorphism ϕ from a group G onto a group G'. Show that the identity e of G is mapped onto the identity e' of G': $e' = \phi(e)$.

Hint: Use the fact that $e = ee$ must be preserved by ϕ and that $\phi(g) = e'\phi(g)$ for all g in G.

Group Theory

Solutions to Problem Set 3 November 2, 2001

1. According to Lagrange's theorem, the order of a subgroup H of a group G must be a divisor of $|G|$. Since the divisors of a prime number are only the number itself and unity, the subgroups of a group of prime order must be either the unit element alone, $H = \{e\}$, or the group G itself, $H = G$, both of which are *improper* subgroups. Therefore, a group of prime order has no *proper* subgroups.

2. From a group G of prime order, select any element g, which is not the unit element, and form its period:
$$g, g^2, g^3, \ldots, g^n = e\,,$$
where n is the order of g (Sec. 2.4). The period *must* include every element in G, because otherwise we would have constructed a subgroup whose order is neither unity nor $|G|$. This contradicts the conclusion of Problem 1. Hence, a group of prime order is necessarily cyclic (but a cyclic group need not necessarily be of prime order).

3. For a group G with $|G| = pq$, where p and q are both prime, we know from Lagrange's theorem that the only proper subgroups have order p and q. Since these subgroups are of prime order, the conclusion of Problem 2 requires these subgroups to be cyclic.

4. Since the period of an element g of a group G forms a subgroup of G (this is straightforward to verify), Lagrange's theorem requires that $|g|$ must be a divisor of $|G|$, i.e., $G = k|g|$ for some integer k. Hence,
$$g^{|G|} = g^{k|g|} = (g^{|g|})^k = e^k = e\,.$$

5. The identity e of a group G has the property that for every element g in G, $ag = ge = g$. We also have that different cosets either have no common elements or have only common elements. Thus, in the factor group G/H of G generated by a subgroup H, the set which contains the unit element corresponds to the unit element of the factor group, since

$$\{e, h_1, h_2, \ldots\}\{a, b, c, \ldots\} = \{a, b, c, \ldots\}\,.$$

6. The class of an element a in a group G is defined as the set of elements gag^{-1} for all elements g in G. If G is Abelian, then we have

$$gag^{-1} = gg^{-1}a = a$$

for all g in G. Hence, in an Abelian group, every element is in a class by itself.

7. Let H be a subgroup of a group of G of index 2, i.e., H has two left cosets and two right cosets. If H is self-conjugate, then $gHg^{-1} = H$ for any g in G. Therefore, to show that H is self-conjugate, we must show that $gH = Hg$ for any g in G, i.e., that the left and right cosets are the same. Since H has index 2, and H is itself a right coset, all of the elements in Hg must either be in H or in the other coset of H, which we will call A. There two possibilities: either g is in H or g is not in H. If g is an element of H, then, according to the Rearrangement Theorem,

$$Hg = gH = H\,.$$

If g is not in H, then it is in A, which is a right coset of H. Two (left or right) cosets of a subgroup have either all elements in common or no elements in common. Thus, since the unit element must be contained in H, the set Hg will contain g which, by hypothesis, is in A. We conclude that

$$Hg = gH = A\,.$$

Therefore,

$$Hg = gH$$

for all g in G and H is, therefore, a self-conjugate subgroup.

8. The subgroup $H = \{e, a^2\}$ of the group $G = \{e, a, a^2, a^3, a^4 = e\}$ has index 2. Therefore, according to Problem 7, H *must* be self-conjugate. Therefore, the elements of the factor group G/H are the subgroup H, which corresponds to the unit element, so we call it $\mathcal{E}$, and the set consisting of the elements $\mathcal{A} = \{a, a^3\}$: $G/H = \{\mathcal{E}, \mathcal{A}\}$.

9. Let ϕ be an isomorphism between a group G and a group G', i.e. ϕ is a one-to-one mapping between all the elements g of G and g' of G'. From the group properties we have that the identity e of G must obey the relation

$$e = ee\,.$$

Since ϕ preserves all products, this relation must in particular be preserved by ϕ:

$$\phi(e) = \phi(e)\phi(e)\,.$$

The group properties require that, for any element g of G,

$$\phi(g) = e'\phi(g)\,,$$

where e' is the identity of G'. Setting $g = e$ and comparing with the preceding equation yields the equality

$$\phi(e)\phi(e) = e'\phi(e)\,,$$

which, by cancellation, implies

$$e' = \phi(e)\,.$$

Thus, an isomorphism maps the identity in G onto the identity in G'.

Group Theory

Problem Set 4 October 30, 2001

Note: Problems marked with an asterisk are for Rapid Feedback.

1.* Given a set of matrices $D(g)$ that form a representation a group G, show that the matrices which are obtainable by a similarity transformation $UD(g)U^{-1}$ are also a representation of G.

2.* Show that the trace of three matrices A, B, and C satisfies the following relation:

$$\mathrm{tr}(ABC) = \mathrm{tr}(CAB) = \mathrm{tr}(BCA)$$

3. Generalize the result in Problem 4 to show that the trace of an n-fold product of matrices is invariant under cyclic permutations of the product.

4.* Show that the trace of an arbitrary matrix A is invariant under a similarity transformation UAU^{-1}.

5. Consider the following representation of S_3:

$$e = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad a = \frac{1}{2}\begin{pmatrix} 1 & -\sqrt{3} \\ -\sqrt{3} & -1 \end{pmatrix}, \quad b = \frac{1}{2}\begin{pmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix}$$

$$c = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad d = \frac{1}{2}\begin{pmatrix} -1 & -\sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix}, \quad f = \frac{1}{2}\begin{pmatrix} -1 & \sqrt{3} \\ -\sqrt{3} & -1 \end{pmatrix}$$

How can these matrices be permuted to provide an equally faithful representation of S_3? Relate your result to the class identified with each element.

6.* Consider the planar symmetry operations of an equilateral triangle. Using the matrices in Example 3.2 determined from transformations of the coordinates in Fig. 3.1, construct a *three*-dimensional representation of S_3 in the (x, y, z) coordinate system, where the z-axis emanates from the geometric center of the triangle. Is this representation reducible or irreducible? If it is reducible determine the irreducible representations which form the direct sum of this representation.

7. Show that two matrices are simultaneously diagonalizable if and only if they commute.

Hint: Two matrices A and B are simultaneously diagonalizable if the *same* similarity transformation brings both matrices into a form where they have only diagonal entries. Proving that simultaneous diagonalizability implies commutativity is straightforward. To prove the converse, suppose that there is a similarity transformation which brings *one* of the matrices into diagonal form. By writing out the matrix elements of the products and using the fact that A and B commute, show that the same similarity transformation *must* also diagonalize the other matrix.

8.* What does the result of Problem 7 imply about the dimensionalities of the irreducible representations of Abelian groups?

9.* Verify that the matrices

$$e = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad a = \tfrac{1}{2}\begin{pmatrix} -1 & \sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix}$$

form a representation for the two-element group $\{e, a\}$. Is this representation reducible or irreducible? If it is reducible determine the one-dimensional representations which form the direct sum of this representation.

10.* Prove the relations in Eqns (3.9) and (3.11).

Group Theory

Solutions to Problem Set 4 November 9, 2001

1. Let $D'(g) = UD(g)U^{-1}$, where $D(g)$ is a representation of a group G with elements g. To show that $D'(g)$ is also a representation of G, it is sufficient to show that this representation preserves the multiplication table of G. Thus, let a and b be any two elements of G with matrix representations $D(a)$ and $D(b)$. The product ab is represented by

$$D(ab) = D(a)D(b)\,.$$

Therefore,

$$\begin{aligned} D'(ab) &= UD(ab)U^{-1} \\ &= UD(a)D(b)U^{-1} \\ &= UD(a)U^{-1}UD(b)U' \\ &= D'(a)D'(b)\,, \end{aligned}$$

so multiplication is preserved and $D'(g)$ is therefore also a representation of G.

2. The trace of a matrix is the sum of its diagonal elements. Therefore, the trace of the product of three matrices A, B, and C is given by

$$\mathrm{tr}(ABC) = \sum_{ijk} A_{ij}B_{jk}C_{ki}\,.$$

By using the fact that i, j, and k are dummy summation indices with the same range, this sum can be written in the equivalent forms

$$\sum_{ijk} A_{ij}B_{jk}C_{ki} = \sum_{ijk} C_{ki}A_{ij}B_{jk} = \sum_{ijk} B_{jk}C_{ki}A_{ij}\,.$$

But the second and third of these are

$$\sum_{ijk} C_{ki}A_{ij}B_{jk} = \mathrm{tr}(CAB)$$

and

$$\sum_{ijk} B_{jk}C_{ki}A_{ij} = \mathrm{tr}(BCA)\,,$$

respectively. Thus, we obtain the relation

$$\mathrm{tr}(ABC) = \mathrm{tr}(CAB) = \mathrm{tr}(BCA)\,.$$

3. The trace of an n-fold product, $A_1A_2\cdots A_n$ is

$$\mathrm{tr}(A_1A_2\cdots A_n) = \sum_{i_1,i_2,\ldots i_n} (A_1)_{i_1i_2}(A_2)_{i_2i_3}\cdots(A_n)_{i_ni_1}\,.$$

Proceeding as in Problem 2, we observe that the i_k ($k = 1, \ldots, n$) are dummy summation indices all of which have the same range. Thus, any cyclic permutation of the matrices in the product leaves the sum and, hence, the trace invariant.

4. From Problem 2, we have that

$$\mathrm{tr}(UAU^{-1}) = \mathrm{tr}(U^{-1}UA) = \mathrm{tr}(A)\,,$$

so a similarity transformation leaves the trace of a matrix invariant.

5. Given a faithful representation of a group, similarity transformations of the matrices provide equally faithful representations. Since we wish to obtain permutations of a particular matrix representation, we base our similarity transformations on the non-unit elements in the group. Thus, consider the following similarity transformations:

$$\begin{aligned}
a\{e,a,b,c,d,f\}a^{-1} &= \{e,a,c,b,f,d\}\,, \\
b\{e,a,b,c,d,f\}b^{-1} &= \{e,c,b,a,f,d\}\,, \\
c\{e,a,b,c,d,f\}c^{-1} &= \{e,b,a,c,f,d\}\,, \\
d\{e,a,b,c,d,f\}d^{-1} &= \{e,b,c,a,d,f\}\,, \\
e\{e,a,b,c,d,f\}e^{-1} &= \{e,c,a,b,d,f\}\,.
\end{aligned} \tag{1}$$

Thus, the following permutations of the elements $\{e,a,b,c,d,f\}$ provide equally faithful representations:

$$\begin{aligned}
&\{e,a,c,b,f,d\}\,, \quad \{e,c,b,a,f,d\}\,, \\
&\{e,b,a,c,f,d\}\,, \quad \{e,b,c,a,d,f\}\,, \\
&\{e,c,a,b,d,f\}\,.
\end{aligned}$$

Notice that only elements within the same class can be permuted. For S_3, the classes are $\{e\}$, $\{a,b,c\}$, $\{d,f\}$.

6. In the basis (x,y,z) where x and y are given in Fig. 3.1 and the z-axis emanates from the origin of this coordinate system (the geometric center of the triangle), all of the symmetry operations of the equilateral triangle leave the z-axis invariant. This is because the z-axis is either an axis of rotation (for operations d and f) or lies within the reflection plane (for operations a, b, and c). Hence,

the matrices of these operations are given by

$$e = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad a = \tfrac{1}{2}\begin{pmatrix} 1 & -\sqrt{3} & 0 \\ -\sqrt{3} & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix},$$

$$b = \tfrac{1}{2}\begin{pmatrix} 1 & \sqrt{3} & 0 \\ \sqrt{3} & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \qquad c = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$d = \tfrac{1}{2}\begin{pmatrix} -1 & -\sqrt{3} & 0 \\ \sqrt{3} & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \quad f = \tfrac{1}{2}\begin{pmatrix} -1 & \sqrt{3} & 0 \\ -\sqrt{3} & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

This representation is seen to be *reducible* and that it is the direct sum of the representation in Example 3.2 (which, as discussed in Example 3.4, is irreducible) and the identical representation.

7. We must show (i) that two matrices which are simultaneously diagonalizable commute and (ii) that two matrices which commute are simultaneously diagonalizable. Showing (i) is straightforward. For two $d \times d$ matrices A and B which are simultaneously diagonalizable, there is a matrix U such that

$$UAU^{-1} = D_A \qquad \text{and} \qquad UBU^{-1} = D_B\,,$$

where D_A and D_B are diagonal forms of these matrices. Clearly, therefore, we have that

$$D_A D_B = D_B D_A\,.$$

Hence, transforming back to the original basis,

$$\underbrace{(U^{-1}D_AU)}_{A}\underbrace{(U^{-1}D_BU)}_{B} = \underbrace{(U^{-1}D_BU)}_{B}\underbrace{(U^{-1}D_AU)}_{A},$$

so A and B commute.

Now suppose that A and B commute and there is a transformation that brings *one* of these matrices, say A, into the diagonal form D_A:

$$UAU^{-1} = D_A\,.$$

Then, with

$$UBU^{-1} = B'\,,$$

the commutation relation $AB = BA$ transforms to

$$D_A B' = B' D_A\,.$$

The (i,j)th matrix element of these products is

$$\begin{aligned}(D_A B')_{ij} &= \sum_k (D_A)_{ik}(B')_{kj} = (D_A)_{ii}(B')_{ij} \\ &= (B' D_A)_{ij} = \sum_k (B')_{ik}(D_A)_{kj} = (B')_{ij}(D_A)_{jj}\,.\end{aligned}$$

After a simple rearrangement, we have

$$(B')_{ij}[(D_A)_{ii} - (D_A)_{jj}] = 0\,.$$

There are three cases to consider:

Case I. All of the diagonal entries of D_A are distinct. Then,

$$(D_A)_{ii} - (D_A)_{jj} \neq 0 \qquad \text{if } i \neq j\,,$$

so *all* of the off-diagonal matrix elements of B' vanish, i.e., B' is a diagonal matrix. Thus, the same similarity transformation which diagonalizes A also diagonalizes B.

Case II. All of the diagonal entries of D_A are the same. In this case D_A is proportional to the unit matrix, $D_A = cI$, for some complex constant c. Hence, this matrix is *always* diagonal,

$$U(cI)U^{-1} = cI$$

and, in particular, it is diagonal when B is diagonal.

Case III. Some of the diagonal entries are the same and some are distinct. If we arrange the elements of D_A such that the first p elements are the same, $(D_A)_{11} = (D_A)_{22} = \cdots = (D_A)_{pp}$, then D_A has the general form

$$D_A = \begin{pmatrix} cI_p & 0 \\ 0 & D'_A \end{pmatrix},$$

where I_p is the $p \times p$ unit matrix and c is a complex constant. From Cases I and II, we deduce that B must be of the form

$$B = \begin{pmatrix} B_p & 0 \\ 0 & D'_B \end{pmatrix},$$

where B_p is some $p \times p$ matrix and D'_B is a diagonal matrix. Let V_p be the matrix which diagonalizes B:

$$V_p B_p V_p^{-1} = D''_B \,.$$

Then the matrix

$$V = \begin{pmatrix} V_p & 0 \\ 0 & I_{d-p} \end{pmatrix}$$

diagonalizes B while leaving D_A unchanged. Here, I_{d-p} is the $(d-p) \times (d-p)$ unit matrix.

Hence, in all three cases, we have shown that the same transformation which diagonalizes A also diagonalizes B.

8. The matrices of any representation $\{A_1, A_2, \ldots, A_n\}$ of an Abelian group G commute:

$$A_i A_j = A_j A_i$$

for all i and j. Hence, according to Problem 7, these matrices can all be simultaneously diagonalized. Since this is true of *all* representations of G, we conclude that all irreducible representations of Abelian groups are one-dimensional, i.e., they are numbers with ordinary multiplication as the composition law.

9. To verify that the matrices

$$e = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad a = \tfrac{1}{2}\begin{pmatrix} -1 & \sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix} \tag{2}$$

form a representation for the two-element group $\{e, a\}$, we need to demonstrate that the multiplication table for this group,

	e	a
e	e	a
a	a	e

is fulfilled by these matrices. The products $e^2 = e$, $ea = a$, and $ae = a$ can be verified by inspection. The product a^2 is

$$a^2 = \tfrac{1}{4}\begin{pmatrix} -1 & \sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix}\begin{pmatrix} -1 & \sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = e\,,$$

so the matrices in (2) form a representation of the two-element group.

Since these matrices commute, they can be diagonalized simultaneously (Problem 7). Since the matrix is the unit matrix, we can determine the diagonal form of a, simply by finding its eigenvalues. The characteristic equation of a is

$$\begin{aligned} det(a - \lambda I) &= \begin{vmatrix} -\frac{1}{2} - \lambda & \frac{1}{2}\sqrt{3} \\ \frac{1}{2}\sqrt{3} & \frac{1}{2} - \lambda \end{vmatrix} \\ &= -(\tfrac{1}{2} - \lambda)(\tfrac{1}{2} + \lambda) - \tfrac{3}{4} = \lambda^2 - 1\,. \end{aligned}$$

which yields $\lambda = \pm 1$. Therefore, diagonal form of a is

$$a = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{3}$$

so this representation is the direct sum of the identical representation $\{1, 1\}$, and the "parity" representation $\{1, -1\}$. Note that, according to Problem 8, *every* representation of the two-element group with dimensionality greater than two *must* be reducible.

10. The relations in (3.9) and (3.11) can be proven simultaneously, since they differ only by complex conjugation, which preserves the order of matrices. The (i, j)th matrix element of n-fold product of matrices $A_1, A_2, \ldots, A_n$ is

$$(A_1 A_2 \cdots A_n)_{ij} = \sum_{k_1, k_2, \ldots, k_{n-1}} (A_1)_{ik_1} (A_2)_{k_1 k_2} \cdots (A_n)_{k_{n-1} j}\,.$$

The corresponding matrix element of the transpose of this product is

$$[(A_1 A_2 \cdots A_n)^{\mathrm{t}}]_{ij} = (A_1 A_2 \cdots A_n)_{ji}\,.$$

Thus, since the k_i are dummy indices,

$$\begin{aligned} [(A_1 A_2 \cdots A_n)^{\mathrm{t}}]_{ij} &= \sum_{k_1, k_2, \ldots, k_{n-1}} (A_1)_{jk_1} (A_2)_{k_1 k_2} \cdots (A_n)_{k_{n-1} i} \\ &= \sum_{k_1, k_2, \ldots, k_{n-1}} (A_1^{\mathrm{t}})_{k_1 j} (A_2^{\mathrm{t}})_{k_2 k_1} \cdots (A_n^{\mathrm{t}})_{ik_{n-1}} \\ &= \sum_{k_1, k_2, \ldots, k_{n-1}} (A_n^{\mathrm{t}})_{ik_{n-1}} (A_{n-1}^{\mathrm{t}})_{k_{n-1} k_{n-2}} \cdots (A_2^{\mathrm{t}})_{k_2 k_1} (A_1^{\mathrm{t}})_{k_1 j} \end{aligned}$$

We conclude that

$$(A_1 A_2 \cdots A_n)^{\mathrm{t}} = A_n^{\mathrm{t}} A_{n-1}^{\mathrm{t}} \cdots A_1^{\mathrm{t}}$$

and, similarly, that

$$(A_1 A_2 \cdots A_n)^\dagger = A_n^\dagger A_{n-1}^\dagger \cdots A_1^\dagger$$

Group Theory

Problem Set 5 November 6, 2001

Note: Problems marked with an asterisk are for Rapid Feedback.

1. In proving Theorem 3.2, we established the relation $B_iB_i^\dagger = I$. Using the definitions in that proof, show that this result implies that $B_i^\dagger B_i = I$ as well.

Hint: Show that $B_iB_i^\dagger = I$ implies that $\tilde{A}_i D \tilde{A}_i^\dagger = D$.

2.* Consider the three-element group $G = \{e, a, b\}$ (Sec. 2.4).

(a) Show that this group is Abelian and cyclic (cf. Problem 2, Problem Set 3).

(b) Consider a one-dimensional representation based on choosing $a = z$, where z is a complex number. Show that for this to produce a representation of G, we must require that $z^3 = 1$.

(c) Use the result of (b) to obtain three representations of G. Given what you know about the irreducible representations of Abelian groups (Problem 8, Problem Set 4), are there any other irreducible representations of G?

3.* Generalize the result of Problem 2 to any cylic group of order n.

4.* Use Schur's First Lemma to prove that all the irreducible representations of an Abelian group are one-dimensional.

5.* Consider the following matrices:

$$e = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad a = \tfrac{1}{2}\begin{pmatrix} -1 & -\sqrt{3} \\ -\sqrt{3} & 1 \end{pmatrix}, \quad b = \tfrac{1}{2}\begin{pmatrix} -1 & -\sqrt{3} \\ -\sqrt{3} & 1 \end{pmatrix}$$

$$c = \tfrac{1}{2}\begin{pmatrix} -1 & -\sqrt{3} \\ -\sqrt{3} & 1 \end{pmatrix}, \qquad d = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad f = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Verify that these matrices form a representation of S_3. Use Schur's first Lemma to determine if this representation reducible or irreducible. If reducible, determine the irreducible representations that are obtained from the diagonal form of these matrices.

Group Theory

Solutions to Problem Set 5 November 16, 2001

1. In proving Theorem 3.2, we established that $B_iB_i^\dagger = I$, where

$$B_i = D^{-1/2}\tilde{A}_iD^{1/2}$$

To show that this result implies that $B_i^\dagger B_i = I$, we first use the definitions of B_i and $B_i^\dagger$ to write

$$B_i^\dagger B_i = D^{1/2}\tilde{A}_i^\dagger D^{-1}\tilde{A}_iD^{1/2}$$

We can find an expression for D^{-1} by first rearranging

$$B_iB_i^\dagger = D^{-1/2}\tilde{A}_iD\tilde{A}_i^\dagger D^{-1/2} = I$$

as

$$\tilde{A}_iD\tilde{A}_i^\dagger = D$$

Then, taking the inverse of both sides of this equation yields

$$D^{-1} = \tilde{A}_i^{\dagger -1}D^{-1}\tilde{A}_i^{-1}$$

Therefore,

$$\begin{aligned} B_i^\dagger B_i &= D^{1/2}\tilde{A}_i^\dagger D^{-1}\tilde{A}_iD^{1/2} \\ &= D^{1/2}\tilde{A}_i^\dagger(\tilde{A}_i^{\dagger -1}D^{-1}\tilde{A}_i^{-1})\tilde{A}_iD^{1/2} \\ &= I \end{aligned}$$

2. (a) The multiplication table for the three-element group is shown below (Sec. 2.4):

	e	a	b
e	e	a	b
a	a	b	e
b	b	e	a

We can see immediately that $a^2 = b$ and that $ab = ba = a^3 = e$, Thus, the three-element group can be written as $\{a, a^2, a^3 = e\}$, i.e., it is a cyclic group (and, therefore, Abelian).

(b) By choosing a one-dimensional representation $a = z$, for some complex number z, the multiplication table requires that $a^3 = e$, which means that $z^3 = 1$.

(c) There are three solutions to $z^3 = 1$: $z = 1, \mathrm{e}^{2\pi i/3}, and \mathrm{e}^{4\pi i/3}$. The three irreducible representations are obtained by choosing $a = 1$, $a = \mathrm{e}^{2\pi i/3}$, and $a = \mathrm{e}^{4\pi i/3}$. Denoting these representations by Γ_1, Γ_2, and Γ_3, we obtain

	e	a	b
Γ_1	1	1	1
Γ_2	1	$\mathrm{e}^{2\pi i/3}$	$\mathrm{e}^{4\pi i/3}$
Γ_2	1	$\mathrm{e}^{4\pi i/3}$	$\mathrm{e}^{2\pi i/3}$

3. The preceding Problem can be generalized to any cyclic group of order n. The elements of this group are $\{g, g^2, \ldots, g^n = e\}$. By writing $g = z$, we require that $z^n = 1$. The solutions to this equation are the nth roots of unity:

$$z = \mathrm{e}^{2m\pi i/n}, \quad m = 0, 1, 2, \ldots, n-1$$

Accordingly, there are n irreducible representations based on the n choices $z = \mathrm{e}^{2m\pi i/n}$ together with the requirements of the group multiplication table.

4. Suppose that we have a representation of an Abelian group of dimensionality d is greater than one. Suppose furthermore that these matrices are not all unit matrices (for, if they were, the representation would already be reducible to the d-fold direct sum of the identical representation.) Then, since the group is Abelian, and the representation must reflect this fact, *any* non-unit matrix in the representation commutes with all the other matrices in the representation. According to Schur's First Lemma, this renders the representation reducible. Hence, all the irreducible representations of an Abelian group are one-dimensional.

5. For the following matrices,

$$e = d = f = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad a = b = c = \tfrac{1}{2}\begin{pmatrix} -1 & -\sqrt{3} \\ -\sqrt{3} & 1 \end{pmatrix}$$

to be a representation of S_3, their products must preserve the multiplication table of this group, which was discussed in Sec. 2.4 and is displayed below:

	e	a	b	c	d	f
e	e	a	b	c	d	f
a	a	e	d	f	b	c
b	b	f	e	d	c	a
c	c	d	f	e	a	b
d	d	c	a	b	f	e
f	f	b	c	a	e	d

To determine the multiplication table of this representation, we use the notation E for the unit matrix, corresponding to the elements e, d, and f, and A for the matrix corresponding to the elements a, b, and c. Then, by observing that $A^2 = E$ (as is required by the group multiplication table) the multiplication table of this representation is straightforward to calculate, and is shown below:

	e	a	b	c	d	f
e	E	A	A	A	E	E
a	A	E	E	E	A	A
b	A	E	E	E	A	A
c	A	E	E	E	A	A
d	E	A	A	A	E	E
f	E	A	A	A	E	E

If we now take the multiplication table of S_3 and perform the mapping $\{e, d, f\} \mapsto E$ and $\{a, b, c\} \mapsto A$, we get the same table as that just obtained by calculating the matrix products directly. Hence, these matrices form a representation of S_3.

From Schur's First Lemma, we see that the matrix

$$\tfrac{1}{2}\begin{pmatrix} -1 & -\sqrt{3} \\ -\sqrt{3} & 1 \end{pmatrix}$$

commutes with all the matrices of the representation. Since this is not a unit matrix, the representation must be reducible.

The diagonal form of these matrices must have entries of one-dimensional irreducible representations. Two one-dimensional irreducible representations of S_3 (we will see later that these are the *only* irreducible representations of S_3) are (Example 3.2) the identical representation,

$$A_e = 1, \quad A_a = 1, \quad A_b = 1,$$

$$A_c = 1, \quad A_d = 1, \quad A_f = 1$$

and the 'parity' representation,

$$A_e = 1, \quad A_a = -1, \quad A_b = -1,$$

$$A_c = -1, \quad A_d = 1, \quad A_f = 1$$

Since the diagonal forms of the matrices are obtained by performing a similarity transformation on the original matrices, which preserves the trace, they must take the form

$$e = d = f = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad a = b = c = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

i.e., this reducible representation contains both the identical and parity representations.

Group Theory

Problem Set 6 November 13, 2001

Note: Problems marked with an asterisk are for Rapid Feedback.

1.* Verify the Great Orthogonality Theorem for the following irreducible representation of S_3:

$$e = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad a = \frac{1}{2}\begin{pmatrix} 1 & -\sqrt{3} \\ -\sqrt{3} & -1 \end{pmatrix}, \quad b = \frac{1}{2}\begin{pmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix}$$

$$c = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad d = \frac{1}{2}\begin{pmatrix} -1 & \sqrt{3} \\ -\sqrt{3} & -1 \end{pmatrix}, \quad f = \frac{1}{2}\begin{pmatrix} -1 & -\sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix}$$

2.* Does the following representation of the three-element group $\{e, a, b\}$:

$$e = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad a = \frac{1}{2}\begin{pmatrix} -1 & \sqrt{3} \\ -\sqrt{3} & -1 \end{pmatrix}, \quad b = \frac{1}{2}\begin{pmatrix} -1 & -\sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix}$$

satisfy the Great Orthogonality Theorem? Explain your answer.

3.* Specialize the Great Orthogonality Theorem to Abelian groups. When viewed as the components of a vector in a $|G|$-dimensional space, what does the Great Orthogonality Theorem state about the relationship between different irreducible representations? What bound does this place on the number of irreducible representations of an Abelian group?

4.* Consider the irreducible representations of the three-element calculated in Problem 2 of Problem Set 5.

(a) Verify that the Great Orthogonality Theorem, in the reduced form obtained in Problem 3, is satisfied for these representations.

(b) In view of the discussion in Sec. 4.4, would you expect to find any other irreducible representations of this group?

(c) Would you expect your answer in (b) to apply to cyclic groups of any order?

5.* Consider any Abelian group. By using the notion of the order of an element (Sec. 2.4), determine the *magnitude* of every element in a representation. Is this consistent with the Great Orthogonality Theorem?

Group Theory

Solutions to Problem Set 6 November 23, 2001

1. The Great Orthogonality Theorem states that, for the matrix elements of the same irreducible representation $\{A_1, A_2, \ldots, A_{|G|}\}$ of a group G,

$$\sum_\alpha (A_\alpha)_{ij}(A_\alpha)^*_{i'j'} = \frac{|G|}{d}\delta_{i,i'}\delta_{j,j'}\,.$$

Thus, we first form the vectors $\boldsymbol{V}_{ij}$ whose components are the (i,j)th elements taken from each matrix in the representation in some fixed order. The Great Orthogonality Theorem can then be expressed more concisely as

$$\boldsymbol{V}_{ij} \cdot \boldsymbol{V}^*_{i'j'} = \frac{|G|}{d}\delta_{i,i'}\delta_{j,j'}\,.$$

For the given representation of S_3,

$$e = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad a = \tfrac{1}{2}\begin{pmatrix} 1 & -\sqrt{3} \\ -\sqrt{3} & -1 \end{pmatrix}, \quad b = \tfrac{1}{2}\begin{pmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix},$$

$$c = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad d = \tfrac{1}{2}\begin{pmatrix} -1 & \sqrt{3} \\ -\sqrt{3} & -1 \end{pmatrix}, \qquad f = \tfrac{1}{2}\begin{pmatrix} -1 & -\sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix},$$

these vectors are:

$$\boldsymbol{V}_{11} = \left(1, \tfrac{1}{2}, \tfrac{1}{2}, -1, -\tfrac{1}{2}, -\tfrac{1}{2}\right),$$

$$\boldsymbol{V}_{12} = \left(0, -\tfrac{1}{2}\sqrt{3}, \tfrac{1}{2}\sqrt{3}, 0, \tfrac{1}{2}\sqrt{3}, -\tfrac{1}{2}\sqrt{3}\right),$$

$$\boldsymbol{V}_{21} = \left(0, -\tfrac{1}{2}\sqrt{3}, \tfrac{1}{2}\sqrt{3}, 0, -\tfrac{1}{2}\sqrt{3}, \tfrac{1}{2}\sqrt{3}\right),$$

$$\boldsymbol{V}_{22} = \left(1, -\tfrac{1}{2}, -\tfrac{1}{2}, 1, -\tfrac{1}{2}, -\tfrac{1}{2}\right).$$

Note that, since all of the entries are real, complex conjugation is not required for substitution into the Great Orthogonality Theorem. For $i = j$ and $i' = j'$, with $|G| = 6$ and $d = 2$, we have

$$\boldsymbol{V}_{11} \cdot \boldsymbol{V}_{11} = 1 + \tfrac{1}{4} + \tfrac{1}{4} + 1 + \tfrac{1}{4} + \tfrac{1}{4} = 3\,,$$

$$\boldsymbol{V}_{12} \cdot \boldsymbol{V}_{12} = 0 + \tfrac{3}{4} + \tfrac{3}{4} + 0 + \tfrac{3}{4} + \tfrac{3}{4} = 3\,,$$

$$\boldsymbol{V}_{21} \cdot \boldsymbol{V}_{21} = 0 + \tfrac{3}{4} + \tfrac{3}{4} + 0 + \tfrac{3}{4} + \tfrac{3}{4} = 3\,.$$

$$\boldsymbol{V}_{22} \cdot \boldsymbol{V}_{22} = 1 + \tfrac{1}{4} + \tfrac{1}{4} + 1 + \tfrac{1}{4} + \tfrac{1}{4} = 3\,,$$

all of which are in accord with the Great Orthogonality Theorem. For $i \neq j$ and/or $i' \neq j'$, we have

$$\boldsymbol{V}_{11} \cdot \boldsymbol{V}_{12} = 0 - \tfrac{1}{4}\sqrt{3} + \tfrac{1}{4}\sqrt{3} + 0 - \tfrac{1}{4}\sqrt{3} + \tfrac{1}{4}\sqrt{3} = 0\,,$$

$$\boldsymbol{V}_{11} \cdot \boldsymbol{V}_{21} = 0 - \tfrac{1}{4}\sqrt{3} + \tfrac{1}{4}\sqrt{3} + 0 + \tfrac{1}{4}\sqrt{3} - \tfrac{1}{4}\sqrt{3} = 0\,,$$

$$\boldsymbol{V}_{11} \cdot \boldsymbol{V}_{22} = 1 - \tfrac{1}{4} - \tfrac{1}{4} - 1 + \tfrac{1}{4} + \tfrac{1}{4} = 0\,,$$

$$\boldsymbol{V}_{12} \cdot \boldsymbol{V}_{21} = 0 + \tfrac{3}{4} + \tfrac{3}{4} + 0 - \tfrac{3}{4} - \tfrac{3}{4} = 0\,,$$

$$\boldsymbol{V}_{12} \cdot \boldsymbol{V}_{22} = 0 + \tfrac{1}{4}\sqrt{3} - \tfrac{1}{4}\sqrt{3} + 0 - \tfrac{1}{4}\sqrt{3} + \tfrac{1}{4}\sqrt{3} = 0\,,$$

$$\boldsymbol{V}_{21} \cdot \boldsymbol{V}_{22} = 0 + \tfrac{1}{4}\sqrt{3} - \tfrac{1}{4}\sqrt{3} + 0 + \tfrac{1}{4}\sqrt{3} - \tfrac{1}{4}\sqrt{3} = 0\,,$$

which is also in accord with the Great Orthogonality Theorem.

2. For the following two-dimensional representation of the three-element group,

$$e = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad a = \tfrac{1}{2}\begin{pmatrix} -1 & \sqrt{3} \\ -\sqrt{3} & -1 \end{pmatrix}, \quad b = \tfrac{1}{2}\begin{pmatrix} -1 & -\sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix},$$

we again form the vectors $\boldsymbol{V}_{ij}$ whose components are the (i,j)th elements of each matrix in the representation:

$$\boldsymbol{V}_{11} = \left(1, -\tfrac{1}{2}, -\tfrac{1}{2}\right),$$

$$\boldsymbol{V}_{12} = \left(0, \tfrac{1}{2}\sqrt{3}, -\tfrac{1}{2}\sqrt{3}\right),$$

$$\boldsymbol{V}_{21} = \left(0, -\tfrac{1}{2}\sqrt{3}, \tfrac{1}{2}\sqrt{3}\right),$$

$$\boldsymbol{V}_{22} = \left(1, -\tfrac{1}{2}, -\tfrac{1}{2}\right).$$

Calculating the summation in the Great Orthogonality Theorem, first with $i = j$ and $i' = j'$, we have

$$\boldsymbol{V}_{11} \cdot \boldsymbol{V}_{11} = 1 + \tfrac{1}{4} + \tfrac{1}{4} = \tfrac{3}{2},$$

$$\boldsymbol{V}_{12} \cdot \boldsymbol{V}_{12} = 0 + \tfrac{3}{4} + \tfrac{3}{4} = \tfrac{3}{2},$$

$$\boldsymbol{V}_{21} \cdot \boldsymbol{V}_{12} = 0 + \tfrac{3}{4} + \tfrac{3}{4} = \tfrac{3}{2},$$

$$\boldsymbol{V}_{11} \cdot \boldsymbol{V}_{11} = 1 + \tfrac{1}{4} + \tfrac{1}{4} = \tfrac{3}{2},$$

all of which are in accord with the Great Orthogonality Theorem with $|G| = 3$ and $d = 2$. Performing the analogous summations with $i \neq j$ and/or $i' \neq j'$, yields

$$\boldsymbol{V}_{11} \cdot \boldsymbol{V}_{12} = 0 - \tfrac{1}{4}\sqrt{3} + \tfrac{1}{4}\sqrt{3} = 0,$$

$$\boldsymbol{V}_{11} \cdot \boldsymbol{V}_{21} = 0 + \tfrac{1}{4}\sqrt{3} - \tfrac{1}{4}\sqrt{3} = 0,$$

$$\boldsymbol{V}_{11} \cdot \boldsymbol{V}_{22} = 1 + \tfrac{1}{4} + \tfrac{1}{4} = \tfrac{3}{2},$$

$$\boldsymbol{V}_{12} \cdot \boldsymbol{V}_{21} = 0 - \tfrac{3}{4} - \tfrac{3}{4} = -\tfrac{3}{2},$$

$$\boldsymbol{V}_{12} \cdot \boldsymbol{V}_{22} = 0 - \tfrac{1}{4}\sqrt{3} + \tfrac{1}{4}\sqrt{3} = 0,$$

$$\boldsymbol{V}_{21} \cdot \boldsymbol{V}_{22} = 0 + \tfrac{1}{4}\sqrt{3} - \tfrac{1}{4}\sqrt{3} = 0,$$

which is *not* consistent with the Great Orthogonality Theorem, since *all* of these quantities must vanish. If there is even a single violation of the Great Orthogonality Theorem, as is the case here, the representation is necessarily reducible.

3. All of the irreducible representations of an Abelian group are one-dimensional (e.g., Problem 4, Problem Set 5). Hence, for Abelian groups, the Great Orthogonality Theorem reduces to

$$\sum_{\alpha} A_{\alpha}^{k} A_{\alpha}^{k'*} = |G| \delta_{k,k'} \,.$$

If we view the irreducible representations as $|G|$-dimensional vectors $\boldsymbol{A}^k$ with entries A_α^k,

$$\boldsymbol{A}^k = (A_1^k, A_2^k, \ldots A_{|G|}^k) \,,$$

then the Great Orthogonality Theorem can be written as a "dot" product:

$$\boldsymbol{A}^k \cdot \boldsymbol{A}^{k'*} = |G| \delta_{k,k'} \,.$$

This states that the irreducible representations of an Abelian group are *orthogonal* vectors in this $|G|$-dimensional space. Since there can be at most $|G|$ such vectors, the number of irreducible representations of an Abelian group is less than or equal to the order of the group.

4. (a) From Problem 2 of Problem Set 5, the irreducible representations of the three element group are:

	e	a	b
Γ_1	1	1	1
Γ_2	1	$\mathrm{e}^{2\pi i/3}$	$\mathrm{e}^{4\pi i/3}$
Γ_2	1	$\mathrm{e}^{4\pi i/3}$	$\mathrm{e}^{2\pi i/3}$

In the notation of Problem 3, we have

$$\boldsymbol{A}^1 = (1,1,1), \quad \boldsymbol{A}^2 = (1, \mathrm{e}^{2\pi i/3}, \mathrm{e}^{4\pi i/3}), \quad \boldsymbol{A}^3 = (1, \mathrm{e}^{4\pi i/3}, \mathrm{e}^{2\pi i/3}) \,.$$

Note that some of these entries are complex. Thus, the distinct inner products between these vectors are

$$\boldsymbol{A}^1 \cdot \boldsymbol{A}^{1*} = 1 + 1 + 1 = 3 \,,$$

$$\begin{aligned}
\boldsymbol{A}^2\cdot\boldsymbol{A}^{2*} &= 1+1+1=3\,,\\
\boldsymbol{A}^3\cdot\boldsymbol{A}^{3*} &= 1+1+1=3\,,\\
\boldsymbol{A}^1\cdot\boldsymbol{A}^{2*} &= 1+\mathrm{e}^{-2\pi i/3}+\mathrm{e}^{-4\pi i/3}=0\,,\\
\boldsymbol{A}^1\cdot\boldsymbol{A}^{3*} &= 1+\mathrm{e}^{-4\pi i/3}+\mathrm{e}^{-2\pi i/3}=0\,,\\
\boldsymbol{A}^2\cdot\boldsymbol{A}^{3*} &= 1+\mathrm{e}^{-2\pi i/3}+\mathrm{e}^{2\pi i/3}=0\,,
\end{aligned}$$

all of which are consistent with the Great Orthogonality Theorem.

(b) In view of the fact that there are 3 mutually orthogonal vectors, there can be no additional irreducible representations of this group.

(c) For cyclic groups of order $|G|$, we determined that the irreducible representations were based on the $|G|$th roots of unity (Problem 3, Problem Set 5). Since this produces $|G|$ distinct irreducible representations, our procedure yields *all* of the irreducible representations of any cyclic group.

5. Every irreducible representation of an Abelian group is one-dimensional. Moreover, since every one of these representations is either a homomorphism or isomorphism of the group, with the operation in the representation being ordinary multiplication, the identity always corresponds to unity (Problem 9, Problem Set 3). Now, the order n of a group element g is the smallest integer for which

$$g^n = e\,.$$

For every element in any group $1 \le n \le |G|$. This relationship *must* be preserved by the irreducible representation. Thus, if A_g^k is the entry corresponding to the element g in the kth irreducible representation, then

$$(A_g^k)^n = 1\,,$$

i.e., A_g^k is the nth root of unity:

$$A_g^k = \mathrm{e}^{2m\pi i/n}, \quad m = 0, 1, \ldots, n-1\,.$$

The modulus of each of these quantities is clearly unity, so the modulus of *every* entry in the irreducible representations of an Abelian group is unity.

This is consistent with the Great Orthogonality Theorem when applied to a given representation (cf. Problem 3):

$$\sum_\alpha A_\alpha^k A_\alpha^{k^*} = \sum_\alpha \left| A_\alpha^k \right|^2 = |G|\,. \tag{1}$$

Group Theory

Problem Set 7 November 20, 2001

Note: Problems marked with an asterisk are for Rapid Feedback.

1.* Identify the 12 symmetry operations of a regular hexagon.

2. Show that elements in the same class of a group must have the same order.

3.* Identify the 6 classes of this group.

Hint: You do not need to compute the conjugacy classes explicitly. Refer to the discussion for the group S_3 in Example 2.9 and use the fact that elements in the same class have the same order.

4.* How many irreducible representations are there and what are their dimensions?

5.* Construct the character table of this group by following the procedure outlined below:

(a) Enter the characters for the identical and "parity" representations. As in the case of S_3, the characters for the parity representation are either $+1$ or -1, depending on whether or not the the operation preserves the "handedness" of the coordinate system.

(b) Enter the characters for the "coordinate" representation obtained from the action on (x, y) for each group operation. Note that the character is the same for elements in the same class.

(c) Use the products $C_3C_3^2 = E$ and $C_3^3 = E$ to identify the characters for all one-dimensional irreducible representations for the appropriate classes. The meaning of the notation C_n^m for rotations is discussed in Section 5.4.

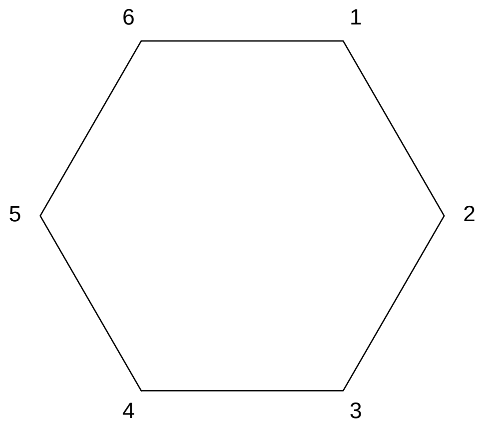
(d) Use the result of (c) and the products $C_6C_3 = C_2$ to deduce that the characters for the class of C_6 and those for the class of C_2 are the same. Then, use the orthogonality of the *columns* of the character table to compute these characters.

(e) Use the appropriate orthogonality relations for characters to compute the remaining entries of the character table.

Group Theory

Solutions to Problem Set 7 November 30, 2001

1. A regular hexagon is shown below:

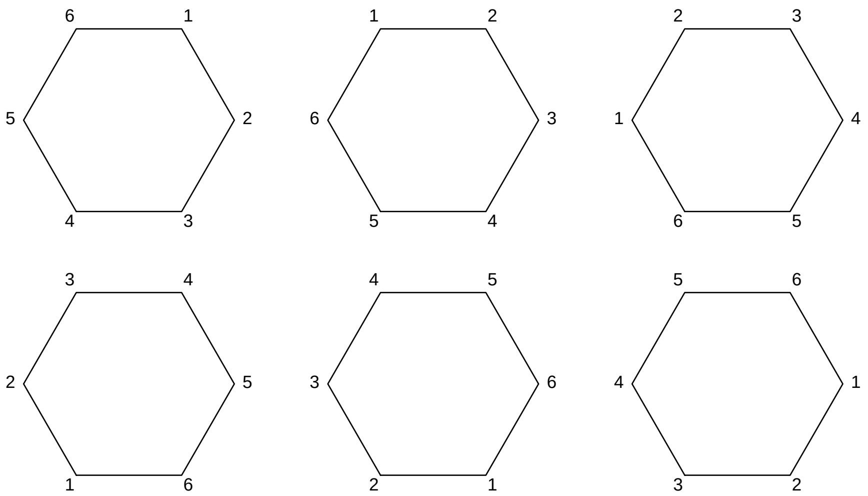


The following notation will be used for the symmetry operations of this hexagon:

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \end{pmatrix},$$

where the first row corresponds to the reference order of the vertices shown in the diagram and the a_i denote the number at the ith vertex *after* the transformation of the hexagon.

The symmetry operations of this hexagon consist of the identity, rotations by angles of $\frac{1}{3}n\pi$ radians, where $n = 1, 2, 3, 4, 5$, three mirror planes which pass through opposite *faces* of the hexagon, and three mirror planes which pass through opposite *vertices* of the hexagon. For the identity and the rotations, the effect on the hexagon is



These operations correspond to

$$E = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 3 & 4 & 5 & 6 \end{pmatrix},$$

$$C_6 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 4 & 5 & 6 & 1 \end{pmatrix},$$

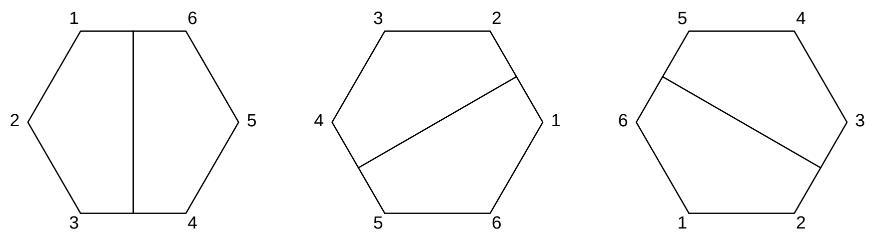
$$C_6^2 = C_3 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 4 & 5 & 6 & 1 & 2 \end{pmatrix},$$

$$C_6^3 = C_2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 1 & 2 & 3 \end{pmatrix},$$

$$C_6^4 = C_3^2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 6 & 1 & 2 & 3 & 4 \end{pmatrix},$$

$$C_6^5 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 1 & 2 & 3 & 4 & 5 \end{pmatrix}.$$

The three mirror planes which pass through opposite *faces* of the hexagon are



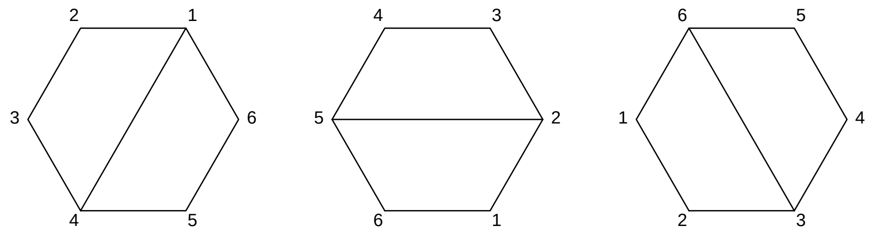
which correspond to

$$\sigma_{v,1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 5 & 4 & 3 & 2 & 1 \end{pmatrix},$$

$$\sigma_{v,2} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 1 & 6 & 5 & 4 & 3 \end{pmatrix},$$

$$\sigma_{v,3} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 3 & 2 & 1 & 6 & 5 \end{pmatrix}.$$

Finally, the three mirror planes which pass through opposite *vertices* of the hexagon are



These operations correspond to

$$\sigma_{d,1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 6 & 5 & 4 & 3 & 2 \end{pmatrix},$$

$$\sigma_{d,2} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 2 & 1 & 6 & 5 & 4 \end{pmatrix},$$

$$\sigma_{d,3} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 4 & 3 & 2 & 1 & 6 \end{pmatrix}.$$

That these 12 elements do, in fact, form a group is straightforward to verify. The standard notation for this group is C_{6v}.

2. The order n of an element a of a group is defined as the smallest integer such that $a^n = e$. If group elements a and b are in the same class, then there is an element g in the group such that

$$b = g^{-1}ag\,.$$

The m-fold product of b is then given by

$$b^m = \underbrace{(g^{-1}ag)(g^{-1}ag)\cdots(g^{-1}ag)}_{m\,\text{factors}} = g^{-1}a^m g\,.$$

If this is equal to the unit element e, we must have

$$g^{-1}a^m g = e\,,$$

or,

$$a^m = e\,.$$

The smallest value of m for which this equality can be satisfied is, by definition, n, the order of a. Hence, two elements in the same class have the same order.

3. For two elements a and b of a group to be in the same class, there must be another group element such that $b = g^{-1}ag$. If the group elements are coordinate transformations, then elements in the same class correspond to the same type of operation, but in coordinate systems related by symmetry operations. This fact, together with the result of Problem 2, allows us to determine the classes of the group of the hexagon.

The identity, as always, is in a class by itself. Although all of the rotations are the same type of operation, not all of these rotations have the same orders: C_6 and C_6^5 have order 6, C_3 and C_3^2 have order 3, and C_2 has order 2. Thus, the 5 rotations belong to *three* different classes.

The two types of mirror planes, $\sigma_{v,i}$ and $\sigma_{d,i}$, *must* belong to different classes since there is no group operation which will transform any of the $\sigma_{v,i}$ to any of the $\sigma_{d,i}$. To do so would require a rotation by an odd multiple of $\frac{1}{6}\pi$, which is not a group element. All of the $\sigma_{v,i}$ are in the same class and all of the $\sigma_{d,i}$ are in the same class, since each is the same type of operation, but in coordinate systems related by symmetry operations (one of the rotations) and, of course, they all have order 2, since each reflection plane is its own inverse.

Hence, there are six classes in this group are

$$\begin{aligned} E &\equiv \{E\}\,, \\ 2C_6 &\equiv \{C_6, C_6^5\}\,, \\ 2C_3 &\equiv \{C_3, C_3^2\}\,, \\ C_2 &\equiv \{C_2\}\,, \\ 3\sigma_v &\equiv \{\sigma_{v,1}, \sigma_{v,2}, \sigma_{v,3}\}\,, \\ 3\sigma_d &\equiv \{\sigma_{d,1}, \sigma_{d,2}, \sigma_{d,3}\}\,. \end{aligned}$$

4. As there are 6 classes, there are 6 irreducible representations, the dimensions of which must satisfy the sum rule

$$\sum_{k=1}^{6} d_k^2 = 12\,,$$

since $|C_{6v}| = 12$. The only positive integer solutions of this equation are

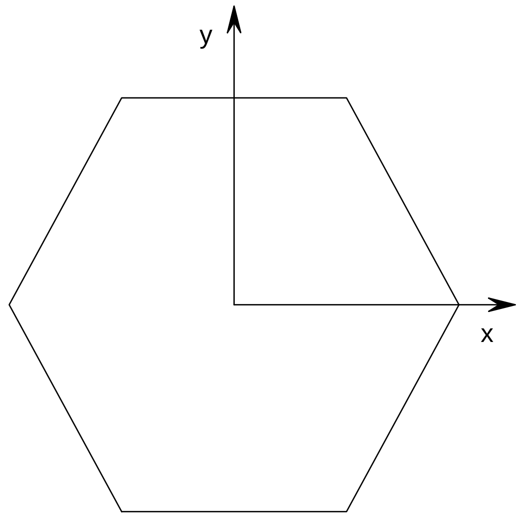
$$d_1 = 1, \quad d_2 = 1, \quad d_3 = 1, \quad d_4 = 1, \quad d_5 = 2, \quad d_6 = 2\,,$$

i.e., there are 4 one-dimensional irreducible representations and 2 two-dimensional irreducible representations.

5. (a) For the identical representation, all of the characters are 1. For the parity representation, the character is 1 for operations which preserve the parity of the coordinate system ("proper" rotations) and -1 for operations which change the parity of the coordinate system ("improper" rotations). Additionally, we can enter immediately the *column* of characters for the class of the unit element. These are equal to the dimensionality of each irreducible representation, since the unit element is the identity matrix with that dimensionality. Thus, we have the following entries for the character table of C_{6v}:

C_{6v}	E	$2C_6$	$2C_3$	C_2	$3\sigma_v$	$3\sigma_d$
Γ_1	1	1	1	1	1	1
Γ_1'	1	1	1	1	-1	-1
Γ_1''	1					
Γ_1'''	1					
Γ_2	2					
Γ_2'	2					

(b) The characters for one of the two-dimensional representations of C_{6v} can be obtained by constructing matrices for operations in analogy with the procedure discussed in Section 3.2 for the equilateral triangle. One important difference here is that we require such a construction only for one element in each class (since the all matrices in a given class have the same trace). We will determine the representations of operations in each class in an (x, y) coordinate system shown below:



Thus, a rotation by an angle ϕ, denoted by $R(\phi)$, is given by the two-dimensional rotational matrix:

$$R(\phi) = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix} .$$

The corresponding character $\chi(\phi)$ is, therefore, simply the sum of the diagonal elements of this matrix:

$$\chi(\phi) = 2\cos\phi \, .$$

We can now calculate the characters for each of the classes composed of rotations:

$$\begin{aligned} \chi(2C_6) &= \chi(\tfrac{1}{3}\pi) = 1 \, , \\ \chi(2C_3) &= \chi(\tfrac{2}{3}\pi) = -1 \, , \\ \chi(C_2) &= \chi(\pi) = -2 \, . \end{aligned}$$

For the two classes of mirror planes, we need only determine the character of one element in each class, which may be chosen at our convenience. Thus, for example, since the representation of

$\sigma_{v,1}$ can be determined directly by inspection:

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix},$$

we can obtain the character of the corresponding class as

$$\chi(3\sigma_v) = 0\,.$$

Similarly, the representation of $\sigma_{d,2}$ can also be determined directly by inspection:

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

which yields the character

$$\chi(3\sigma_d) = 0\,.$$

We can now add the entries for this two-dimensional irreducible representations to the character table of C_{6v}:

C_{6v}	E	$2C_6$	$2C_3$	C_2	$3\sigma_v$	$3\sigma_d$
Γ_1	1	1	1	1	1	1
Γ_1'	1	1	1	1	-1	-1
Γ_1''	1					
Γ_1'''	1					
Γ_2	2	1	-1	-2	0	0
Γ_2'	2					

(c) The one-dimensional irreducible representations must obey the multiplication table, since they themselves are representations of the group. In particular, given the products

$$C_3C_3^2 = E, \qquad C_3^3 = E\,,$$

if we denote by α the character of the class $2C_3 = \{C_3, C_3^2\}$, then these products require that

$$\alpha^2 = 1, \qquad \alpha^3 = 1\,,$$

respectively. Thus, we deduce that $\alpha = 1$ for *all* of the one-dimensional irreducible representations. With these additions to the character table, we have

C_{6v}	E	$2C_6$	$2C_3$	C_2	$3\sigma_v$	$3\sigma_d$
Γ_1	1	1	1	1	1	1
Γ_1'	1	1	1	1	-1	-1
Γ_1''	1		1			
Γ_1'''	1		1			
Γ_2	2	1	-1	-2	0	0
Γ_2'	2					

(d) Since the character for all one-dimensional irreducible representations for the class $2C_3 = \{C_3, C_3^2\}$ is unity, the product $C_6C_3 = C_2$ requires that the characters for the classes of C_6 and C_2 are the same in these representations. Since $C_2^2 = E$, this character must be 1 or -1. Suppose we choose $\chi(2C_6) = \chi(C_2) = 1$. Then, the orthogonality of the *columns* of the character table requires that the character for the classes E and $2C_6$ are orthogonal. If we denote by β the character for the class $2C_6$ of the representation Γ_2', we require

$$(1 \times 1) + (1 \times 1) + (1 \times 1) + (1 \times 1) + (2 \times 1) + (2 \times \beta) = 0\,,$$

i.e., $\beta = -3$. But this value violates the requirement that

$$\sum_\alpha n_\alpha |\chi_\alpha|^2 = |G|\,. \tag{1}$$

Thus, we must choose $\chi(2C_6) = \chi(C_2) = -1$, and our character table becomes

C_{6v}	E	$2C_6$	$2C_3$	C_2	$3\sigma_v$	$3\sigma_d$
Γ_1	1	1	1	1	1	1
Γ_1'	1	1	1	1	-1	-1
Γ_1''	1	-1	1	-1		
Γ_1'''	1	-1	1	-1		
Γ_2	2	1	-1	-2	0	0
Γ_2'	2					

(e) The characters for the classes $2C_6$, $2C_3$, and C_2 of the Γ_2' representation can now be determined by requiring the columns corresponding to these classes to be orthogonal to the column corresponding to the class of the identity. When this is done, we find that the values obtained saturate the sum rule in (1), so the characters corresponding to both classes of mirror planes in this representation must vanish. This enables to complete the entries for the Γ_2' representation:

C_{6v}	E	$2C_6$	$2C_3$	C_2	$3\sigma_v$	$3\sigma_d$
Γ_1	1	1	1	1	1	1
Γ_1'	1	1	1	1	-1	-1
Γ_1''	1	-1	1	-1		
Γ_1'''	1	-1	1	-1		
Γ_2	2	1	-1	-2	0	0
Γ_2'	2	-1	1	2	0	0

The remaining entries are straightforward to calculate. The fact that each mirror reflection has order 2 means that these entries must be either $+1$ or -1. The requirement of orthogonality of columns leaves only one choice:

C_{6v}	E	$2C_6$	$2C_3$	C_2	$3\sigma_v$	$3\sigma_d$
Γ_1	1	1	1	1	1	1
Γ_1'	1	1	1	1	-1	-1
Γ_1''	1	-1	1	-1	1	-1
Γ_1'''	1	-1	1	-1	-1	1
Γ_2	2	1	-1	-2	0	0
Γ_2'	2	-1	1	2	0	0

which completes the character table for C_{6v}.

Group Theory

Problem Set 8 November 27, 2001

Note: Problems marked with an asterisk are for Rapid Feedback.

1. Show that if two matrices A and B are orthogonal, then their direct product $A \otimes B$ is also an orthogonal matrix.

2. Show that the trace of the direct product of two matrices A and B is the product of the traces of A and B:

$$\mathrm{tr}(A \otimes B) = \mathrm{tr}(A)\,\mathrm{tr}(B)$$

3.* Show that the direct product of groups G_a and G_b with elements $G_a = \{e, a_2, \ldots, a_{|G_a|}\}$ and $G_b = \{e, b_2, \ldots, b_{|G_b|}\}$, such that $a_i b_j = b_j a_i$ for all i and j, is a group. What is the order of this group?

4.* Use the Great Orthogonality Theorem to show that the direct product of irreducible representations of two groups is an irreducible representation of the direct product of those groups.

5.* For an n-fold degenerate set of eigenfunctions φ_i, $i, 1, 2, \ldots, n$, we showed show that the matrices $\Gamma(R_\alpha)$ generated by the group of the Hamiltonian,

$$R_\alpha \varphi_i = \sum_{j=1}^{n} \varphi_j \Gamma_{ji}(R_\alpha)$$

form a representation of that group. Show that if the φ_j are chosen to be an orthonormal set of functions, then this representation is *unitary.*

6.* The set of distinct functions obtained from a given function φ_i by operations in the group of the Hamiltonian, $\varphi_j = R_\alpha \varphi_i$, are called **partners**. Use the Great Orthogonality Theorem to show that two functions which belong to different irreducible representations or are different partners in the same unitary representation are orthogonal.

7. Consider a particle of mass m confined to a square in two dimensions whose vertices are located at $(1, 1)$, $(1, -1)$, $(-1, -1)$, and $(-1, 1)$. The potential is taken to be zero within the square and infinite at the edges of the square. The eigenfunctions φ are of the form

$$\varphi_{p,q}(x, y) \propto \left\{ \begin{matrix} \cos(k_p x) \\ \sin(k_p x) \end{matrix} \right\} \left\{ \begin{matrix} \cos(k_q y) \\ \sin(k_q y) \end{matrix} \right\}$$

where $k_p = \frac{1}{2}p\pi$, $k_q = \frac{1}{2}q\pi$, and p and q are positive integers. The notation above means that $\cos(k_p x)$ is taken if p is odd, $\sin(k_p x)$ is taken if p is even, and similarly for the other factor. The corresponding eigenvalues are

$$E_{p,q} = \frac{\hbar^2 \pi^2}{8m}(p^2 + q^2)$$

(a) Determine the eight planar symmetry operations of a square. These operations form the group of the Hamiltonian for this problem. Assemble the symmetry operations into equivalence classes.

(b) Determine the number of irreducible representations and their dimensions for this group. Do these dimensions appear to be broadly consistent with the degeneracies of the energy eigenvalues?

(c) Determine the action of each group operation on (x, y).

Hint: This can be done by inspection.

(d) Determine the characters corresponding to the identical, parity, and coordinate representations. Using appropriate orthogonality relations, complete the character table for this group.

(e) For which irreducible representations do the eigenfunctions $\varphi_{1,1}(x, y)$ and $\varphi_{2,2}(x, y)$ form bases?

(f) For which irreducible transformation do the eigenfunctions $\varphi_{1,2}(x, y)$ and $\varphi_{2,1}(x, y)$ form a basis?

(g) What is the degeneracy corresponding to $(p = 6, q = 7)$ and $(p = 2, q = 9)$? Is this a normal or accidental degeneracy?

(h) Are there eigenfunctions which form a basis for each of the irreducible representations of this group?

8.* Consider the regular hexagon in Problem Set 7. Suppose there is a vector perturbation, i.e., a perturbation that transforms as (x, y, z). Determine the selection rule associated with an initial state that transforms according to the "parity" representation.

Hint: The reasoning for determining the irreducible representations associated with (x, y, z) is analogous to that used in Section 6.6.2 for the equilateral triangle.

Group Theory

Solutions to Problem Set 8 December 7, 2001

1. A matrix A is said to be orthogonal if its matrix elements a_{ij} satisfy the following relations:

$$\sum_i a_{ij}a_{ik} = \delta_{j,k}, \qquad \sum_j a_{ij}a_{kj} = \delta_{ij}\ , \tag{1}$$

i.e., the rows and columns are orthogonal vectors. This ensures that $A^{\rm t}A = AA^{\rm t} = I$.

The direct product C of two matrices A and B, denoted by $C = A \otimes B$, is given in terms of matrix elements by

$$c_{ik;jl} = a_{ij}a_{kl}\ .$$

If A and B are orthogonal matrices, then we can show that C is also an orthogonal matrix by verifying the relations in Eq. (1). The first of these relations is

$$\sum_{ik} c_{ik;jl}c_{ik;j'l'}$$

$$= \sum_{ik} a_{ij}a_{kl}a_{ij'}a_{kl'} = \Big(\sum_i a_{ij}a_{ij'}\Big)\Big(\sum_k a_{kl}a_{kl'}\Big) = \delta_{j,j'}\delta_{l,l'}\ ,$$

where the last step follows from the first of Eqs. (1). The second orthogonality relation is

$$\sum_{jl} c_{ik;jl}c_{i'k';jl}$$

$$= \sum_{jl} a_{ij}a_{kl}a_{i'j}a_{k'l} = \Big(\sum_j a_{ij}a_{i'j}\Big)\Big(\sum_l a_{kl}a_{k'l}\Big) = \delta_{i,i'}\delta_{k,k'}$$

where the last step follows from the second of Eqs. (1). Thus, we have shown that the direct product of two orthogonal matrices is also an orthogonal matrix.

2. The direct product of two matrices A and B with matrix elements a_{ij} and b_{ij} is

$$c_{ik,jl} = a_{ij} b_{kl}\,.$$

The trace of the direct product $A \otimes B$ is obtained by setting $j = i$ and $l = k$ and summing over i and k:

$$\mathrm{tr}(A \otimes B) = \sum_{ik} c_{ik,ik} = \sum_{ik} a_{ii} b_{kk} = \sum_i a_{ii} \sum_k b_{kk} = \mathrm{tr}(A)\,\mathrm{tr}(B)\,,$$

which is the product of the traces of A and B.

3. We have two groups G_a and G_b with elements

$$G_a = \{e_a, a_2, a_3, \ldots, a_{|G_a|}\}$$

and

$$G_b = \{e_b, b_2, b_3, \ldots, b_{|G_b|}\}\,,$$

such that $a_i b_j = b_j a_i$ for all i and j. We are using a notation where it is understood that $a_1 = e_a$ and $b_1 = e_b$. The direct product $G_a \otimes G_b$ of these groups is the set obtained by forming the product of every element of G_a with every element of G_b:

$$G_a \otimes G_b = \{e, a_2, a_3, \ldots, a_{n_a}, b_2, b_3, \ldots, b_{n_b}, \ldots, a_i b_j, \ldots\}\,.$$

To show that $G_a \otimes G_b$ is a group, we must demonstrate that these elements fulfill each of the four requirements in Sec. 2.1.

Closure. The product of two elements $a_i b_j$ and $a_{i'} b_{j'}$ is given by

$$(a_i b_j)(a_{i'} b_{j'}) = (a_i a_{i'})(b_j b_{j'}) = a_k b_l\,,$$

where the first step follows from the commutativity of elements between the two groups and the second step from the group property of G_a and G_b.

Associativity. The associativity of the composition law follows from

$$\begin{aligned}(a_i b_{i'} a_j b_{j'}) a_k b_{k'} &= \Big[(a_i a_j) a_k\Big]\Big[(b_{i'} b_{j'}) b_{k'}\Big] \\ &= \Big[a_i (a_j a_k)\Big]\Big[b_{i'} (b_{j'} b_{k'})\Big] \\ &= a_i b_{i'} (a_j b_{j'} a_k b_{k'})\,,\end{aligned}$$

since associativity holds for G_a and G_b separately.

Unit Element. The unit element e for the direct product group is $e_a e_b = e_b e_a$, since

$$(a_i b_j)(e_a e_b) = (a_i e_a)(b_j e_b) = (e_a a_i)(e_b b_j) = (e_a e_b)(a_i b_j)\,.$$

Inverse. Finally, the inverse of each element $a_i b_j$ is $a_i^{-1} b_j^{-1}$ because

$$(a_i b_j)(a_i^{-1} b_j^{-1}) = (a_i a_i^{-1})(b_j b_j^{-1}) = e_a e_b$$

and

$$(a_i^{-1} b_j^{-1})(a_i b_j) = (a_i^{-1} a_i)(b_j^{-1} b_j) = e_a e_b\,.$$

Thus, we have shown that the direct product of two groups is itself a group. Since the elements of this group are obtained by taking all products of elements from G_a and G_b, the order of this group is $|G_a||G_b|$.

4. Suppose we have an irreducible representation for each of two groups G_a and G_b. We denote these representations, which may be of different dimensions, by $A(a_i)$ and $A(b_j)$, and their matrix elements by $A(a_i)_{ij}$ and $A(b_j)_{ij}$. Since these representations are irreducible, they satisfy the Great Orthogonality Theorem:

$$\sum_{a_i} A(a_i)^*_{ij} A(a_i)_{i'j'} = \frac{|G_a|}{d_a} \delta_{i,i'} \delta_{j,j'}\,,$$

$$\sum_{b_j} A(b_j)^*_{ij} A(b_j)_{i'j'} = \frac{|G_b|}{d_b} \delta_{i,i'} \delta_{j,j'}\,,$$

where d_a and d_b are the dimensions of the irreducible representations of G_a and G_b, respectively. A representation of the direct product of two groups, denoted by $A(a_ib_j)$, is obtained from the direct product of representations of each group:

$$A(a_ib_j)_{ik;jl} = A(a_i)_{ij}A(b_j)_{kl}\,.$$

The sum in the Great Orthogonality Theorem for the direct product representation is

$$\begin{aligned}
&\sum_{a_i}\sum_{b_j} A(a_ib_j)^*_{ik;jl}A(a_ib_j)_{i'k';j'l'} \\
&= \sum_{a_i}\sum_{b_j} A(a_i)^*_{ij}A(b_j)^*_{kl}A(a_i)_{i'j'}A(b_j)_{k'l'} \\
&= \underbrace{\Big[\sum_{a_i} A(a_i)^*_{ij}A(a_i)_{i'j'}\Big]}_{\frac{|G_a|}{d_a}\delta_{i,i'}\delta_{j,j'}}\underbrace{\Big[\sum_{b_j} A(b_j)^*_{kl}A(b_j)_{k'l'}\Big]}_{\frac{|G_b|}{d_b}\delta_{k,k'}\delta_{l,l'}} \\
&= \left(\frac{|G_a||G_b|}{d_ad_b}\right)\delta_{i,i'}\delta_{k,k'}\delta_{j,j'}\delta_{l,l'}\,.
\end{aligned}$$

This shows that this direct product representation is, in fact, irreducible. It has dimensionality d_ad_b and the order of the direct product is, of course, $|G_a| \times |G_b|$.

5. If the φ_i are orthonormal, and if this property is required to be preserved by the group of the Hamiltonian (as it must, to conserve probability), then, in Dirac notation, we have

$$\begin{aligned}
(i,j) \equiv \int \varphi_i(x)^*\varphi_j(x)\,\mathrm{d}x &= \int [R\varphi_i(x)]^* R\varphi_j(x)\,\mathrm{d}x \\
&= (i|R^\dagger R|j) = \delta_{i,j}\,.
\end{aligned}$$

Therefore,

$$\begin{aligned}
(i|R^\dagger R|j) &= \sum_{k,l}(k,l)\Gamma(R)^*_{ki}\Gamma(R)_{lj} \\
&= \sum_k \Gamma(R)^*_{ki}\Gamma(R)_{kj} \\
&= \sum_k \left[\Gamma(R)^\dagger\right]_{ik}\Gamma(R)_{kj} \\
&= \left[\Gamma(R)^\dagger\Gamma(R)\right]_{ij},
\end{aligned}$$

i.e., when written in matrix notation,

$$\Gamma(R)^\dagger\Gamma(R) = I\,.$$

Thus, the matrix representation is unitary.

6. We again use Dirac notation to signify basis functions φ_i and φ_j corresponding to irreducible representations n and n', respectively: $|n,i)$ and $|n',j)$. Then, the operations R in the group of the Hamiltonian applied to these functions yield

$$\begin{aligned}
R|n,i) &= \sum_k \Gamma^{(n)}(R)_{ki}|n,k)\,, \\
R|n',j) &= \sum_l \Gamma^{(n')}(R)_{lj}|n',l)\,.
\end{aligned}$$

Since the operators and their representations are unitary,

$$(n',j|R^\dagger = (n',j|R^{-1} = \sum_l \Gamma^{(n')}(R)^*_{lj}(n',l|\,,$$

we have

$$\begin{aligned}
(n',j|R^{-1}R|n,i) &= (n',j|n,i) \\
&= \sum_{kl}\Gamma^{(n')}(R)^*_{kj}\Gamma^{(n)}(R)_{li}(n',k|n,l)\,.
\end{aligned}$$

If we now sum both sides of this equation over the elements of the group of the Hamiltonian, and invoke the Great Orthogonality Theorem, we obtain

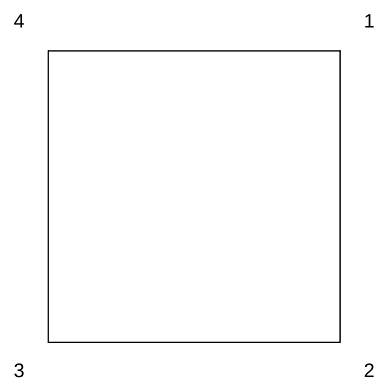
$$\begin{aligned}\sum_R (n',j|n,i) &= |G|(n',j|n,i)\\ &= \sum_{kl}\underbrace{\Big[\sum_R \Gamma^{(n')}(R)^*_{kj}\Gamma^{(n)}(R)_{li}\Big]}_{\frac{|G|}{d_n}\delta_{n,n'}\delta_{k,l}\delta_{i,j}}(n',k|n,l)\\ &= |G|\delta_{n,n'}\delta_{i,j}(n',k|n,k)\,,\end{aligned}$$

where $|G|$ is the order of the group of the Hamiltonian and d_n is the dimension of the nth irreducible representation. Therefore,

$$(n',j|n,i) = \delta_{n,n'}\delta_{i,j}\,,$$

since $(n,k|n,k) = 1$.

7. (a) A square is shown below:



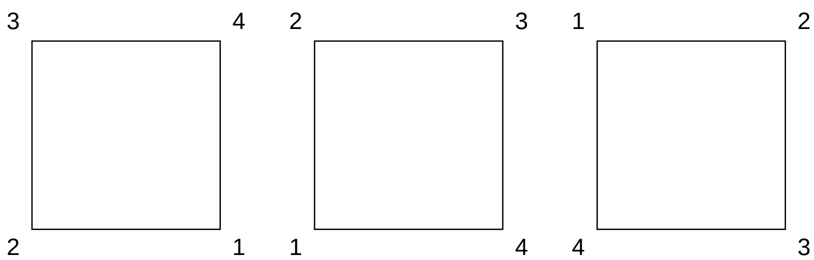
In analogy with the procedure described in Problem Set 7, we will use the following notation for the symmetry operations of this hexagon:

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ a_1 & a_2 & a_3 & a_4 \end{pmatrix},$$

where the a_i denote the number at the ith vertex after the transformation of the hexagon given in the indicated reference order. Thus, the identity operation, which identifies the reference order of the vertices, corresponds to

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}.$$

The symmetry operations on this square consist of the identity, rotations by angles of $\frac{1}{2}n\pi$ radians, for $n = 1, 2, 3$, two mirror planes which pass through opposite *faces* of the square, and two mirror planes which pass through opposite *vertices* of the square. For the rotations, the effect on the square is



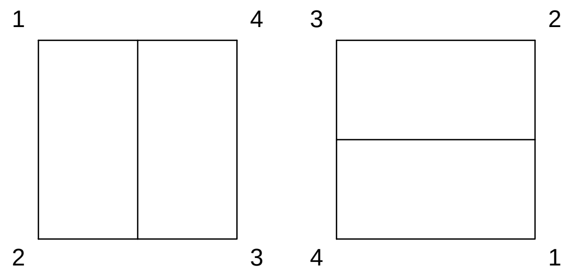
These operations correspond to

$$C_4 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix},$$

$$C_4^2 = C_2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix},$$

$$C_4^3 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}.$$

The two mirror planes which pass through opposite *faces* of the square are

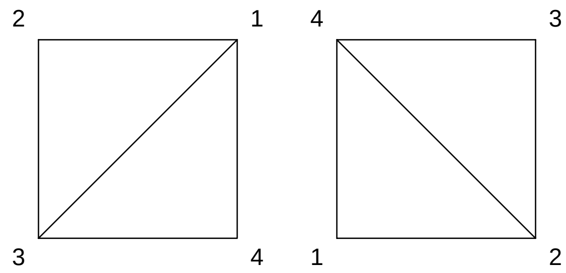


which correspond to

$$\sigma_{v,1} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix} ,$$

$$\sigma_{v,2} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix} .$$

Finally, the two mirror planes which pass through opposite *vertices* of the square are



These operations correspond to

$$\sigma'_{v,1} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 3 & 2 \end{pmatrix} ,$$

$$\sigma'_{v,2} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 1 & 4 \end{pmatrix} .$$

Elements in the same equivalence class must have the same order and correspond to the same "type" of operation. Thus, there are *five* equivalence classes of this group:

$$\{E\}, \quad \{C_2 = C_4^2\}, \quad \{2C_4\}, \quad \{2\sigma_v\}, \quad \{2\sigma_v'\}\,.$$

Note that, as in the case of the regular hexagon (Problem Set 7), all of the rotations need not belong to the same class, despite being the same "type" of operation because they must also have the same order.

(b) The order of this group is 8 and there are 5 equivalence classes. Thus, there must be five irreducible representations whose dimensionalities must satisfy

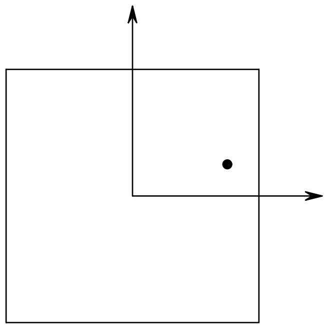
$$d_1^2 + d_2^2 + d_3^2 + d_4^2 + d_5^2 = 8\,.$$

The *only* solution of this equation (with positive integer values for the d_k) is

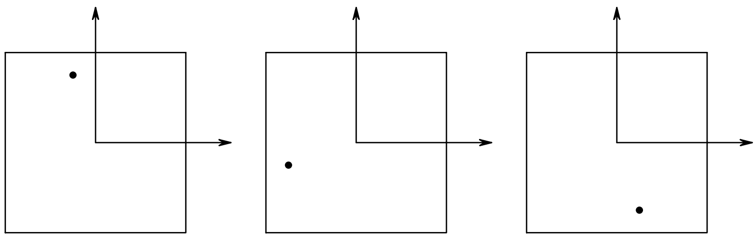
$$d_1 = 1, \quad d_2 = 1, \quad d_3 = 1, \quad d_4 = 1, \quad d_5 = 2\,.$$

These dimensionalities imply that the energy levels for a Hamiltonian with this symmetry are either nondegenerate or are two-fold degenerate. From the expression given for the energy eigenvalues, we see immediately that the energy eigenvalues with $p = q$ are non-degenerate, and those with $p \neq q$ are two-fold degenerate (but see below). Thus, the dimensions of the irreducible representations are consistent with these degeneracies.

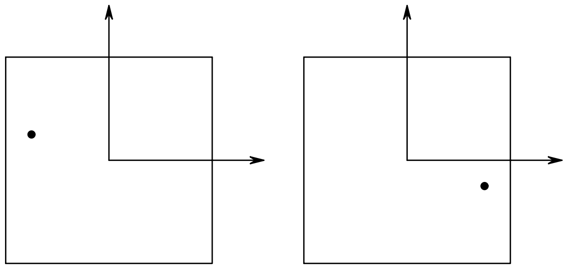
(c) The simplest way to obtain a two-dimensional representation of this group is to consider the action of each group element on some generic point (x, y). Then the action on this point of each of the operations given above can be determined by inspection. We begin with the figure below:



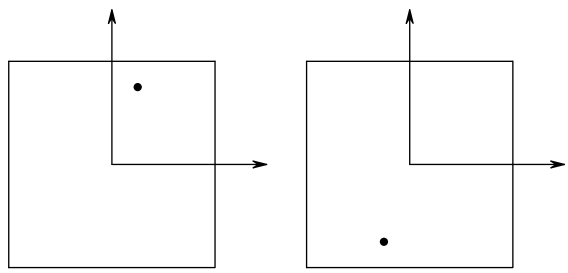
From the diagrammatic representation of each symmetry operation, we will be able to determine the corresponding representation, simply by inspection. The action on this point by the three rotations acn be represented as



These rotations are thus seen to transform the point (x, y) into $(-y, x)$, $(-x, -y)$, and $(y, -x)$, respectively. The two reflections that pass through the center of faces are



so they transform the (x, y) into $(-x, y)$ and $(x, -y)$, respectively. Finally, for the two reflection planes which pass through vertices,



which transform the point (x, y) into (y, x) and $(-y, -x)$, respectively. These transformations enable us to construct the characters corresponding to the "coordinate" representation. Then, together with the identical and parity representations, we have the following entries of the character table for this group:

	E	C_2	$2C_4$	$2\sigma_v$	$2\sigma'_v$
Γ_1	1	1	1	1	1
Γ'_1	1	1	1	-1	-1
Γ''_1	1				
Γ'''_1	1				
Γ_2	2	-2	0	0	0

The group multiplication table and the orthogonality of *columns* allows us to immediately complete the entries for the classes $\{C_2\}$ and $\{2C_4\}$:

	E	C_2	$2C_4$	$2\sigma_v$	$2\sigma'_v$
Γ_1	1	1	1	1	1
Γ'_1	1	1	1	-1	-1
Γ''_1	1	1	-1		
Γ'''_1	1	1	-1		
Γ_2	2	-2	0	0	0

The remaining four entries can be determined from the orthogonality of either rows or columns and again invoking the group multiplication table:

	E	C_2	$2C_4$	$2\sigma_v$	$2\sigma_v'$
Γ_1	1	1	1	1	1
Γ_1'	1	1	1	-1	-1
Γ_1''	1	1	-1	1	-1
Γ_1'''	1	1	-1	-1	1
Γ_2	2	-2	0	0	0

(e) The eigenfunctions $\varphi_{1,1}(x,y)$ and $\varphi_{2,2}(x,y)$ are given by

$$\varphi_{1,1}(x,y) \propto \cos(\tfrac{1}{2}\pi x)\cos(\tfrac{1}{2}\pi y)$$

and

$$\varphi_{2,2}(x,y) \propto \sin(\pi x)\sin(\pi y)\,.$$

Since $\varphi_{1,1}(x,y)$ is invariant under the interchange of x and y and under changes in their signs, it transforms according to the **identical** representation. However, although $\varphi_{2,2}(x,y)$ is invariant under the interchange of x and y, each sine factor changes sign if their argument changes sign. Thus, this eigenfunction transforms according to the **parity** representation.

(f) The (degenerate) eigenfunctions $\varphi_{1,2}(x,y)$ and $\varphi_{2,1}(x,y)$ are

$$\begin{pmatrix} \varphi_{1,2} \\ \varphi_{2,1} \end{pmatrix} \propto \begin{bmatrix} \cos(\tfrac{1}{2}\pi x)\sin(\pi y) \\ \sin(\pi x)\cos(\tfrac{1}{2}\pi y) \end{bmatrix}.$$

The transformation properties of these eigenfunctions can be determined from the results of part (c). This yields the following matrix representation of each symmetry operation:

$$E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad C_4 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad C_4^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix},$$

$$C_4^3 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_{v,1} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \sigma_{v,2} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$\sigma_{v,1}' = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_{v,2}' = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}.$$

This produces the following characters:

$$\{E\} = 2, \quad \{C_2\} = -2, \quad \{2C_4\} = 0, \quad \{2\sigma_v\} = 0, \quad \{2\sigma_v'\} = 0\,,$$

which are the characters of the two-dimensional irreducible representation Γ_2, which is the "coordinate" irreducible representation.

(g) We have that the energies $E_{6,7}$ and $E_{2,9}$ are given by

$$E_{6,7} = E_{7,6} = \frac{\hbar^2\pi^2}{8m}(6^2 + 7^2) = 85\frac{\hbar^2\pi^2}{8m}$$

and

$$E_{2,9} = E_{9,2} = \frac{\hbar^2\pi^2}{8m}(2^2 + 9^2) = 85\frac{\hbar^2\pi^2}{8m}\,,$$

so this energy is *fourfold* degenerate. However, since the group operations have the effect of interchanging x and y with possible changes of sign, the eigenfunctions $\varphi_{6,7}$ and $\varphi_{7,6}$ are transformed only between one another, and the eigenfunctions $\varphi_{2,9}$ and $\varphi_{9,2}$ are transformed only between one another. In other words, this fourfold degeneracy is **accidental**, resulting only from the numerical coincidence of the energies of two twofold-degenerate states.

(h) We have already determined that $\varphi_{p,p}$ with p even transforms according to the identical representation, while if p is odd, $\varphi_{p,p}$ transforms according to the parity representation. Moreover, the pair of eigenfunctions $\varphi_{p,q}$ where p is even and q is odd transforms according to the coordinate representation.

Consider now the case where the eigenfunctions are of the form $\varphi_{p,q}$ where *both* p and q are even. The matrices corresponding to the symmetry operations are

$$E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad C_4 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad C_4^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$C_4^3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_{v,1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_{v,2} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$\sigma'_{v,1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma'_{v,2} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The corresponding characters are

$$\{E\} = 2, \quad \{C_2\} = 2, \quad \{2C_4\} = 0, \quad \{2\sigma_v\} = 2, \quad \{2\sigma'_v\} = 0\,.$$

This representation must be *reducible*, since its characters do not correspond to those of any of the irreducible representations in the table determined in Part (d). A straightforward application of the Decomposition Theorem (or simple inspection) shows that this representation is the direct sum of the Γ_1 and Γ''_1 irreducible representations. This means that there is a linear combination of these eigenfunctions that diagonalizes the matrices corresponding to each symmetry operation of this group.

For the eigenfunctions of the form $\varphi_{p,q}$ where *both* p and q are odd, the matrices corresponding to the symmetry operations are

$$E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad C_4 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \quad C_4^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$C_4^3 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \quad \sigma_{v,1} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_{v,2} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix},$$

$$\sigma'_{v,1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma'_{v,2} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The characters are now

$$\{E\} = 2, \quad \{C_2\} = 2, \quad \{2C_4\} = 0, \quad \{2\sigma_v\} = -2, \quad \{2\sigma'_v\} = 0\,.$$

which correspond to a *reducible* representation composed of the direct sum of the Γ_1' and Γ_1''' irreducible representations. Thus, *all* of the irreducible representations occur in the eigenfunctions of the two-dimensional square well.

8. The character table of the regular hexagon is reproduced below:

C_{6v}	E	$2C_6$	$2C_3$	C_2	$3\sigma_v$	$3\sigma_d$
Γ_1	1	1	1	1	1	1
Γ_1'	1	1	1	1	-1	-1
Γ_1''	1	-1	1	-1	1	-1
Γ_1'''	1	-1	1	-1	-1	1
Γ_2	2	1	-1	-2	0	0
Γ_2'	2	-1	1	2	0	0

A transformation properties of a vector perturbation can be deduced in a manner analogous to that for the equilateral triangle (Section 6.6.2). Applying each symmetry operation to $\boldsymbol{r} = (x, y, z)$ produces a *reducible* representation because these operations are either rotations or reflections through vertical planes. Thus, the z axis is invariant under every symmetry operation of this group which. Together with the fact that an (x, y) basis generates the two-dimensional irreducible representation Γ_2 [Problem 5(b), Problem Set 7], yields

$$\Gamma' = \Gamma_1 \oplus \Gamma_2 \,.$$

The corresponding characters are

C_{6v}	E	$2C_6$	$2C_3$	C_2	$3\sigma_v$	$3\sigma_d$
$\Gamma_1 \oplus \Gamma_2$	3	2	0	-1	1	1

to determine the selection rule for an initial state that transforms according to the parity representation (Γ_1', we must calculated

$$\Gamma_1' \otimes \Gamma' = \Gamma_1' \otimes (\Gamma_1 \oplus \Gamma_2) \,.$$

The characters associated with this operation are

C_{6v}	E	$2C_6$	$2C_3$	C_2	$3\sigma_v$	$3\sigma_d$
$\Gamma_1' \otimes (\Gamma_1 \oplus \Gamma_2)$	3	2	0	-1	-1	-1

Finally, either by inspection, or by applying the decomposition theorem, we find that

$$\Gamma_1' \otimes (\Gamma_1 \oplus \Gamma_2) = \Gamma_1' \oplus \Gamma_2 \,,$$

so transitions between states that transform according to the parity representation and any states other than those that transform as the parity or coordinate representations are forbidden by symmetry.

Group Theory

Problem Set 9 December 4, 2001

Note: Problems marked with an asterisk are for Rapid Feedback.

1.* Consider the group $\mathrm{O}(n)$, the elements of which preserve the Euclidean length in n dimensions:

$$x_1'^2 + x_2'^2 + \cdots + x_n'^2 = x_1^2 + x_2^2 + \cdots + x_n^2 \,.$$

Show that these transformations have $\frac{1}{2}n(n-1)$ free parameters.

2. The condition that the Euclidean length is preserved in two dimensions, $x'^2 + y'^2 = x^2 + y^2$, was shown in lectures to require that

$$a_{11}^2 + a_{21}^2 = 1, \qquad a_{11}a_{12} + a_{21}a_{22} = 0, \qquad a_{12}^2 + a_{22}^2 = 1 \,.$$

Show that these requirements imply that

$$(a_{11}a_{22} - a_{12}a_{21})^2 = 1 \,.$$

3. Rotations in two dimensions can be parametrized by

$$R(\varphi) = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix} .$$

Show that

$$R(\varphi_1 + \varphi_2) = R(\varphi_1)R(\varphi_2)$$

and, hence, deduce that this group is Abelian.

4.* We showed in lectures that a rotation $R(\varphi)$ by an angle φ in two dimensions can be written as

$$R(\varphi) = \mathrm{e}^{\varphi X} \,,$$

where

$$X = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} .$$

Verify that

$$\mathrm{e}^{\varphi X} = I\cos\varphi + X\sin\varphi \,,$$

where I is the two-dimensional unit matrix. This shows that $\mathrm{e}^{\varphi X}$ is the rotation matrix in two dimensions.

5.* Consider the two-parameter group

$$x' = ax + b$$

Determine the infinitesimal operators of this group.

6.* Consider the group $C_{\infty v}$ which contains, in addition to all two-dimensional rotations, a reflection plane, denoted by σ_v in, say, the x-z plane. Is this group Abelian? What are the equivalence classes of this group?

Hint: Denoting reflection in the x–z plane by S, show that $SR(\varphi)S^{-1} = R(-\varphi)$.

7. By extending the procedure used in lectures for SO(3), show that the infinitesimal generators of SO(4), the group of proper rotations in four dimensions which leave the quantity $x^2 + y^2 + z^2 + w^2$ invariant, are

$$A_1 = z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z}, \qquad A_2 = x\frac{\partial}{\partial z} - z\frac{\partial}{\partial x}, \qquad A_3 = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}$$

$$B_1 = x\frac{\partial}{\partial w} - w\frac{\partial}{\partial x}, \qquad B_2 = y\frac{\partial}{\partial w} - w\frac{\partial}{\partial y}, \qquad B_3 = z\frac{\partial}{\partial w} - w\frac{\partial}{\partial z}$$

8. Show that the commutators of the generators obtained in Problem 7 are

$$[A_i, A_j] = \varepsilon_{ijk}A_k, \qquad [B_i, B_j] = \varepsilon_{ijk}A_k, \qquad [A_i, B_j] = \varepsilon_{ijk}B_k$$

9. Show that by making the linear transformation of the generators in Problem 7 to

$$J_i = \tfrac{1}{2}(A_i + B_i), \qquad K_i = \tfrac{1}{2}(A_i - B_i)$$

the commutators become

$$[J_i, J_j] = \varepsilon_{ijk}J_k, \qquad [K_i, K_j] = \varepsilon_{ijk}K_k, \qquad [J_i, K_j] = 0$$

This shows that locally SO(4)=SO(3)$\otimes$SO(3).

Group Theory

Solutions to Problem Set 9 December 14, 2001

1. The Lie group GL(n,R) has n^2 parameters, because the transformations can be represented as $n \times n$ matrices (with real entries). The requirement that the Euclidean length dimensions be preserved by such a transformation leads to the requirement that,

$$x_1'^2 + x_2'^2 + \cdots + x_n'^2 = x_1^2 + x_2^2 + \cdots + x_n^2\,.$$

Proceeding as in Sec. 7.2, we note that there are n conditions from the requirement that the coefficients of x_i, $i = 1, 2, \ldots, n$ be equal to unity. Then, there are $\frac{1}{2}n(n-1)$ conditions from the requirement that the coefficients of the *unique* products $x_i x_j$, $i \neq j$ vanish. Thus, beginning with n free parameters for GL(n,R), there are

$$n^2 - n - \tfrac{1}{2}n(n-1) = n^2 - n - \tfrac{1}{2}n^2 + \tfrac{1}{2}n = \tfrac{1}{2}n(n-1)$$

free parameters for O(n).

2. Beginning with the three conditions

$$a_{11}^2 + a_{21}^2 = 1, \qquad a_{11}a_{12} + a_{21}a_{22} = 0, \qquad a_{12}^2 + a_{22}^2 = 1\,,$$

we take the product of the first by the third equations and subtract the square of the second equation to obtain

$$\begin{aligned}
&(a_{11}^2 + a_{21}^2)(a_{12}^2 + a_{22}^2) - (a_{11}a_{12} + a_{21}a_{22})^2 \\
&= a_{11}^2 a_{12}^2 + a_{11}^2 a_{22}^2 + a_{21}^2 a_{12}^2 + a_{21}^2 a_{22}^2 - a_{11}^2 a_{12}^2 \\
&\qquad -2a_{11}a_{12}a_{21}a_{22} - a_{21}^2 a_{22}^2 \\
&= a_{11}^2 a_{22}^2 + a_{21}^2 a_{12}^2 - 2a_{11}a_{12}a_{21}a_{22} \\
&= (a_{11}a_{22} - a_{12}a_{21})^2 \\
&= 1\,.
\end{aligned}$$

Thus, the three constraints for orthogonal groups in two dimensions imply that the square of the determinant of such transformation must be equal to unity.

3. Forming the product of the the matrices corresponding to $R(\varphi_1)$ and $R(\varphi_2)$ yields

$$\begin{pmatrix} \cos\varphi_1 & -\sin\varphi_1 \\ \sin\varphi_1 & \cos\varphi_1 \end{pmatrix} \begin{pmatrix} \cos\varphi_2 & -\sin\varphi_2 \\ \sin\varphi_2 & \cos\varphi_2 \end{pmatrix}$$

$$= \begin{pmatrix} \cos\varphi_1\cos\varphi_2 - \sin\varphi_1\sin\varphi_2 & -\cos\varphi_1\sin\varphi_2 - \sin\varphi_1\cos\varphi_2 \\ \sin\varphi_1\cos\varphi_2 + \cos\varphi_1\sin\varphi_2 & -\sin\varphi_1\sin\varphi_2 + \cos\varphi_1\cos\varphi_2 \end{pmatrix}.$$

By invoking the standard trigonometric identities for the sines and cosines of the sum and difference of two angles,

$$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y\,,$$
$$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y\,,$$

we can write

$$\begin{pmatrix} \cos\varphi_1 & -\sin\varphi_1 \\ \sin\varphi_1 & \cos\varphi_1 \end{pmatrix} \begin{pmatrix} \cos\varphi_2 & -\sin\varphi_2 \\ \sin\varphi_2 & \cos\varphi_2 \end{pmatrix}$$

$$= \begin{bmatrix} \cos(\varphi_1+\varphi_2) & -\sin(\varphi_1+\varphi_2) \\ \sin(\varphi_1+\varphi_2) & \cos(\varphi_1+\varphi_2) \end{bmatrix}.$$

Thus,

$$R(\varphi_1+\varphi_2) = R(\varphi_1)R(\varphi_2)\,.$$

4. The expression

$$R(\varphi) = \mathrm{e}^{\varphi X}\,,$$

where

$$X = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\,,$$

is defined by its Taylor series:

$$\mathrm{e}^{\varphi X} = \sum_{n=0}^{\infty} \frac{1}{n!}(\varphi X)^n\,. \tag{1}$$

Successive powers of X yield

$$X = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad X^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\,,$$

$$X^3 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad X^4 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\,,$$

whereupon this sequence is repeated. We can write this sequence in matrix form as $X^2 = -I, X^3 = -X, X^4 = I, \ldots$, where I is the 2×2 unit matrix. The powers of X are therefore given by

$$X^{2n} = \begin{cases} I, & n \text{ even} \\ -I, & n \text{ odd} \end{cases}$$

for even powers and

$$X^{2n+1} = \begin{cases} X, & n \text{ even} \\ -X, & n \text{ odd} \end{cases}$$

for odd powers Thus, the Taylor series in (1) may be written as

$$\mathrm{e}^{\varphi X} = \underbrace{\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}\varphi^{2n}}_{\cos\varphi} I + \underbrace{\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\varphi^{2n+1}}_{\sin\varphi} X$$

$$= I\cos\varphi + X\sin\varphi$$

$$= \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix},$$

which is the rotation matrix in two dimensions.

5. The two parameter group

$$x' = ax + b$$

was discussed in Example 7.1. The identity was found to correspond to the parameters $a = 1$ and $b = 0$. The infinitesimal transformations are therefore given by

$$x' = (1 + \mathrm{d}a)x + \mathrm{d}b = x + x\,\mathrm{d}a + \mathrm{d}b\,.$$

If we substitute this into some function $f(x)$ and expand to first order in the parameters a and b, we obtain

$$f(x') = f(x + x\,\mathrm{d}a + \mathrm{d}b) = f(x) + x\frac{\partial f}{\partial x}\,\mathrm{d}a + \frac{\partial f}{\partial x}\,\mathrm{d}b\,,$$

from which we identify the infinitesimal operators

$$X_1 = x\frac{\partial}{\partial x}, \qquad X_2 = \frac{\partial}{\partial x}\,.$$

6. The group $C_{\infty v}$ contains all two-dimensional rotations and a vertical reflection plane, denoted by σ_v, in the x-z plane. Since this reflection changes the parity of the coordinate system, it changes the sense of the rotation angle φ. Thus, a rotation by φ in the original coordinate system corresponds to a rotation by $-\varphi$ in the

transformed coordinate system. Denoting the reflection operator by S, we then must have that

$$SR(\varphi)S^{-1} = R(-\varphi)\,. \tag{2}$$

Since $S = S^{-1}$, we can see this explicitly for the two-dimensional rotation matrix $R(\varphi)$:

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix},$$

where the matrix on the right-hand side of this equation is $R(-\varphi)$. Equation (2) shows that (i) the group is no longer Abelian, and (ii) the equivalence classes correspond to rotations by φ and $-\varphi$.

7. Proceeding as in Section 7.4, the infinitesimal rotations in four dimensions which leave the quantity $x^2 + y^2 + z^2 + w^2$ invariant are

$$\begin{pmatrix} x' \\ y' \\ z' \\ w' \end{pmatrix} = \begin{pmatrix} 1 & \varphi_1 & \varphi_2 & \varphi_3 \\ -\varphi_1 & 1 & \varphi_4 & \varphi_5 \\ -\varphi_2 & -\varphi_4 & 1 & \varphi_6 \\ -\varphi_3 & -\varphi_5 & -\varphi_6 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}.$$

Substituting this coordinate transformation into a differentiable function $F(x, y, z, w)$,

$$\begin{aligned} &F(x', y', z', w') \\ &= F(x + \varphi_1 y + \varphi_2 z + \varphi_3 w, y - \varphi_1 x - \varphi_4 z + \varphi_5 w, \\ &\qquad z - \varphi_2 x + \varphi_4 y + \varphi_6 w, w - \varphi_3 x - \varphi_5 y - \varphi_6 z)\,. \end{aligned}$$

and expanding to first order in the φ_i, yields

$$F(x', y', z', w') = F(x, y, z, w) + \varphi_1\Big(y\frac{\partial F}{\partial x} - x\frac{\partial F}{\partial y}\Big)$$

$$= \varphi_2\Big(z\frac{\partial F}{\partial x} - x\frac{\partial F}{\partial z}\Big) + \varphi_3\Big(w\frac{\partial F}{\partial x} - x\frac{\partial F}{\partial w}\Big)$$

$$= \varphi_4\Big(y\frac{\partial F}{\partial z} - z\frac{\partial F}{\partial y}\Big) + \varphi_5\Big(w\frac{\partial F}{\partial y} - y\frac{\partial F}{\partial w}\Big)$$

$$= \varphi_6\Big(w\frac{\partial F}{\partial z} - z\frac{\partial F}{\partial w}\Big) .$$

From these equations and, if necessary, a change in sign of the corresponding φ_i, we can identify the following differential operators

$$A_1 = z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z}\,, \qquad A_2 = x\frac{\partial}{\partial z} - z\frac{\partial}{\partial x}\,, \qquad A_3 = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}\,,$$

$$B_1 = x\frac{\partial}{\partial t} - t\frac{\partial}{\partial x}\,, \qquad B_2 = y\frac{\partial}{\partial t} - t\frac{\partial}{\partial y}\,, \qquad B_3 = z\frac{\partial}{\partial t} - t\frac{\partial}{\partial z}\,.$$

8. With the infinitesimal generators calculated in Problem 7, we determine the commutators in the standard fashion. For the commutators between the A_i, we have

$$[A_1, A_2]f = A_1(A_2 f) - A_2(A_1 f)$$

$$= \Big(z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z}\Big)\Big(x\frac{\partial f}{\partial z} - z\frac{\partial f}{\partial x}\Big) - \Big(x\frac{\partial}{\partial z} - z\frac{\partial}{\partial x}\Big)\Big(z\frac{\partial f}{\partial y} - y\frac{\partial f}{\partial z}\Big)$$

$$= xz\frac{\partial^2 f}{\partial y \partial z} - z^2\frac{\partial^2 f}{\partial y \partial x} - xy\frac{\partial^2 f}{\partial z^2} + y\frac{\partial f}{\partial x} + yz\frac{\partial^2 f}{\partial z \partial x}$$

$$-x\frac{\partial f}{\partial y} - xz\frac{\partial^2 f}{\partial z \partial y} + xy\frac{\partial^2 f}{\partial z^2} + z^2\frac{\partial^2 f}{\partial x \partial y} + yz\frac{\partial^2 f}{\partial x \partial z}$$

$$= y\frac{\partial f}{\partial x} - -x\frac{\partial f}{\partial y}$$

$$= A_3 f\,,$$

$$[A_1, A_3]f = A_1(A_3 f) - A_3(A_1 f)$$
$$= \Big(z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z}\Big)\Big(y\frac{\partial f}{\partial x} - x\frac{\partial f}{\partial y}\Big) - \Big(y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}\Big)\Big(z\frac{\partial f}{\partial y} - y\frac{\partial f}{\partial z}\Big)$$
$$= z\frac{\partial f}{\partial x} + yz\frac{\partial^2 f}{\partial y \partial z} - xz\frac{\partial^2 f}{\partial y^2} - y^2\frac{\partial^2 f}{\partial z \partial x} + xy\frac{\partial^2 f}{\partial z \partial y}$$
$$-yz\frac{\partial^2 f}{\partial x \partial y} + y^2\frac{\partial^2 f}{\partial x \partial z} + xz\frac{\partial^2 f}{\partial y^2} - x\frac{\partial f}{\partial z} - xy\frac{\partial^2 f}{\partial y \partial z}$$
$$= z\frac{\partial f}{\partial x} - x\frac{\partial f}{\partial z}$$
$$= -A_2 f\,,$$

$$[A_2, A_3]f = A_2(A_3 f) - A_3(A_2 f)$$
$$= \Big(x\frac{\partial}{\partial z} - z\frac{\partial}{\partial x}\Big)\Big(y\frac{\partial f}{\partial x} - x\frac{\partial f}{\partial y}\Big) - \Big(y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}\Big)\Big(x\frac{\partial f}{\partial z} - z\frac{\partial f}{\partial x}\Big)$$
$$= xy\frac{\partial^2 f}{\partial z \partial x} - x^2\frac{\partial^2 f}{\partial z \partial y} - yz\frac{\partial^2 f}{\partial x^2} + z\frac{\partial f}{\partial y} + xz\frac{\partial^2 f}{\partial x \partial y}$$
$$-y\frac{\partial f}{\partial z} - xy\frac{\partial^2 f}{\partial x \partial z} + yz\frac{\partial^2 f}{\partial x^2} + x^2\frac{\partial^2 f}{\partial y \partial z} - xz\frac{\partial^2 f}{\partial y \partial x}$$
$$= z\frac{\partial f}{\partial y} - y\frac{\partial f}{\partial z}$$
$$= A_1 f\,.$$

Thus, we can summarize these results as

$$[A_i, A_j] = \varepsilon_{ijk} A_k\,.$$

Similarly, for the B_i, we calculate the pertinent commutators as

$$[B_1, B_2]f = B_1(B_2 f) - B_2(B_1 f)$$

$$= \left(x\frac{\partial}{\partial w} - w\frac{\partial}{\partial x}\right)\left(y\frac{\partial f}{\partial w} - w\frac{\partial f}{\partial y}\right) - \left(y\frac{\partial}{\partial w} - w\frac{\partial}{\partial y}\right)\left(x\frac{\partial f}{\partial w} - w\frac{\partial f}{\partial x}\right)$$

$$= xy\frac{\partial^2 f}{\partial w^2} - x\frac{\partial f}{\partial y} - xw\frac{\partial^2 f}{\partial w\partial y} - yw\frac{\partial^2 f}{\partial x\partial w} + w^2\frac{\partial^2 f}{\partial x\partial y}$$

$$-xy\frac{\partial^2 f}{\partial w^2} + y\frac{\partial f}{\partial x} + yw\frac{\partial^2 f}{\partial w\partial x} + xw\frac{\partial^2 f}{\partial y\partial t} - w^2\frac{\partial^2 f}{\partial y\partial x}$$

$$= y\frac{\partial f}{\partial x} - x\frac{\partial f}{\partial y}$$

$$= A_3 f\,,$$

$$[B_1, B_3]f = B_1(B_3 f) - B_3(B_1 f)$$

$$= \left(x\frac{\partial}{\partial w} - w\frac{\partial}{\partial x}\right)\left(z\frac{\partial f}{\partial w} - w\frac{\partial f}{\partial z}\right) - \left(z\frac{\partial}{\partial w} - w\frac{\partial}{\partial z}\right)\left(x\frac{\partial f}{\partial w} - w\frac{\partial f}{\partial x}\right)$$

$$= xz\frac{\partial^2 f}{\partial w^2} - x\frac{\partial f}{\partial z} - xw\frac{\partial^2 f}{\partial w\partial z} - zw\frac{\partial^2 f}{\partial x\partial w} + t^2\frac{\partial^2 f}{\partial x\partial z}$$

$$-xz\frac{\partial^2 f}{\partial w^2} + z\frac{\partial f}{\partial x} + zw\frac{\partial^2 f}{\partial w\partial x} + xt\frac{\partial^2 f}{\partial z\partial w} - w^2\frac{\partial^2 f}{\partial z\partial x}$$

$$= z\frac{\partial f}{\partial x} - x\frac{\partial f}{\partial z}$$

$$= -A_2 f\,,$$

$$[B_2, B_3]f = B_2(B_3 f) - B_3(B_2 f)$$

$$= \left(y\frac{\partial}{\partial w} - w\frac{\partial}{\partial y}\right)\left(z\frac{\partial f}{\partial w} - w\frac{\partial f}{\partial z}\right) - \left(z\frac{\partial}{\partial w} - w\frac{\partial}{\partial z}\right)\left(y\frac{\partial f}{\partial w} - w\frac{\partial f}{\partial y}\right)$$

$$= yz\frac{\partial^2 f}{\partial w^2} - y\frac{\partial f}{\partial z} - yw\frac{\partial^2 f}{\partial w\partial z} - zw\frac{\partial^2 f}{\partial y\partial w} + w^2\frac{\partial^2 f}{\partial y\partial z}$$

$$-yz\frac{\partial^2 f}{\partial w^2}+z\frac{\partial f}{\partial y}+zw\frac{\partial^2 f}{\partial w\partial y}+yt\frac{\partial^2 f}{\partial z\partial w}-w^2\frac{\partial^2 f}{\partial z\partial y}$$

$$= z\frac{\partial f}{\partial y}-y\frac{\partial f}{\partial z}$$

$$= A_1 f\,.$$

These results can be summarized as

$$[B_i, B_j] = \varepsilon_{ijk} A_k\,.$$

Finally, for the commutators between the A_i and B_j, we note first by inspection that

$$[A_i, B_i] = 0\,,$$

for $i = 1, 2, 3$, since A_i and B_i involve mutually exclusive pairs of variables. For the remaining commutator pairs, we have

$$[A_1, B_2] = A_1(B_2 f) - B_2(A_1 f)$$

$$= \left(z\frac{\partial}{\partial y}-y\frac{\partial}{\partial z}\right)\left(y\frac{\partial f}{\partial w}-w\frac{\partial f}{\partial y}\right)-\left(y\frac{\partial}{\partial w}-w\frac{\partial}{\partial y}\right)\left(z\frac{\partial f}{\partial y}-y\frac{\partial f}{\partial z}\right)$$

$$= z\frac{\partial f}{\partial w}+yz\frac{\partial^2 f}{\partial y\partial w}-zw\frac{\partial^2 f}{\partial y^2}-y^2\frac{\partial^2 f}{\partial z\partial w}+yw\frac{\partial^2 f}{\partial z\partial y}$$

$$-yz\frac{\partial^2 f}{\partial w\partial y}+y^2\frac{\partial^2 f}{\partial w\partial z}+zw\frac{\partial^2 f}{\partial y^2}-w\frac{\partial f}{\partial z}-yw\frac{\partial^2 f}{\partial y\partial z}$$

$$= z\frac{\partial f}{\partial w}-w\frac{\partial f}{\partial z}$$

$$= B_3 f\,,$$

$$[A_1, B_3] = A_1(B_3 f) - B_3(A_1 f)$$

$$= \left(z\frac{\partial}{\partial y}-y\frac{\partial}{\partial z}\right)\left(z\frac{\partial f}{\partial w}-w\frac{\partial f}{\partial z}\right)-\left(z\frac{\partial}{\partial w}-w\frac{\partial}{\partial z}\right)\left(z\frac{\partial f}{\partial y}-y\frac{\partial f}{\partial z}\right)$$

$$= z^2\frac{\partial^2 f}{\partial y\partial w} - zw\frac{\partial^2 f}{\partial y\partial z} - y\frac{\partial f}{\partial w} - yz\frac{\partial^2 f}{\partial z\partial w} + yw\frac{\partial^2}{\partial z^2}$$

$$-z^2\frac{\partial^2 f}{\partial w\partial y} + yz\frac{\partial^2 f}{\partial w\partial z} + w\frac{\partial f}{\partial y} + zw\frac{\partial^2 f}{\partial z\partial y} - yw\frac{\partial^2 f}{\partial z^2}$$

$$= w\frac{\partial f}{\partial y} - y\frac{\partial f}{\partial w}$$

$$= -B_2 f\,,$$

$$[A_2, B_3] = A_2(B_3 f) - B_3(A_2 f)$$

$$= \left(x\frac{\partial}{\partial z} - z\frac{\partial}{\partial x}\right)\left(z\frac{\partial f}{\partial w} - w\frac{\partial f}{\partial z}\right) - \left(z\frac{\partial}{\partial w} - w\frac{\partial}{\partial z}\right)\left(x\frac{\partial f}{\partial z} - z\frac{\partial f}{\partial x}\right)$$

$$= x\frac{\partial f}{\partial w} + xz\frac{\partial^2 f}{\partial z\partial w} - xw\frac{\partial^2 f}{\partial z^2} - z^2\frac{\partial^2 f}{\partial x\partial w} + zw\frac{\partial^2 f}{\partial x\partial z}$$

$$-xz\frac{\partial^2 f}{\partial w\partial z} + z^2\frac{\partial^2 f}{\partial w\partial x} - w\frac{\partial f}{\partial x} + xw\frac{\partial^2 f}{\partial z^2} - zw\frac{\partial^2 f}{\partial x\partial z}$$

$$= x\frac{\partial f}{\partial w} - w\frac{\partial f}{\partial x}$$

$$= B_1 f\,.$$

Thus,

$$[A_i, B_j] = \varepsilon_{ijk} B_k\,.$$

9. Consider the following linear combinations of the operators in Problem 7:

$$J_i = \tfrac{1}{2}(A_i + B_i), \qquad K_i = \tfrac{1}{2}(A_i - B_i)\,. \tag{3}$$

We can now use the commutation relations derived in Problem 8 to derive the commutation relations for the J_i and K_j. For the J_i, we have

$$\begin{aligned}
[J_i, J_j] &= \tfrac{1}{4}[A_i + B_i, A_j + B_j] \\
&= \tfrac{1}{4}\Big([A_i, A_j] + [A_i, B_j] + [B_i, A_j] + [B_i, B_j]\Big) \\
&= \tfrac{1}{4}(\varepsilon_{ijk}A_k + \varepsilon_{ijk}B_k + \varepsilon_{ijk}B_k + \varepsilon_{ijk}A_k) \\
&= \varepsilon_{ijk}\tfrac{1}{2}(A_k + B_k) \\
&= \varepsilon_{ijk}J_k\,,
\end{aligned}$$

$$\begin{aligned}
[K_i, K_j] &= \tfrac{1}{4}[A_i - B_i, A_j - B_j] \\
&= \tfrac{1}{4}\Big([A_i, A_j] - [A_i, B_j] - [B_i, A_j] + [B_i, B_j]\Big) \\
&= \tfrac{1}{4}(\varepsilon_{ijk}A_k - \varepsilon_{ijk}B_k - \varepsilon_{ijk}B_k + \varepsilon_{ijk}A_k) \\
&= \varepsilon_{ijk}\tfrac{1}{2}(A_k - B_k) \\
&= \varepsilon_{ijk}K_k\,,
\end{aligned}$$

$$\begin{aligned}
[J_i, K_j] &= \tfrac{1}{4}[A_i + B_i, A_j - B_j] \\
&= \tfrac{1}{4}\Big([A_i, A_j] - [A_i, B_j] + [B_i, A_j] - [B_i, B_j]\Big) \\
&= \tfrac{1}{4}(\varepsilon_{ijk}A_k + \varepsilon_{ijk}B_k - \varepsilon_{ijk}B_k - \varepsilon_{ijk}A_k) \\
&= 0\,.
\end{aligned}$$

Group Theory

Problem Set 10 December 11, 2001

Note: Problems marked with an asterisk are for Rapid Feedback.

1.* Prove that a proper orthogonal transformation in an odd-dimensional space always possesses an axis, i.e., a line whose point are left unchanged by the transformation.

2. Prove **Euler's theorem**: The general displacement of a rigid body with one fixed point is a rotation about an axis.

3.* The functions $(x \pm iy)^m$, where m is an integer generate irreducible representations of SO(2). Suppose we now consider the group O(2), where we now allow *improper* rotations. Use Schur's lemma to show that these functions generate irreducible *two*-dimensional representations of O(2) for $m \neq 0$, but a *reducible* representation for $m = 0$.

Hint: The general improper rotation in two dimensions is

$$\begin{pmatrix} \cos\varphi & \sin\varphi \\ \sin\varphi & -\cos\varphi \end{pmatrix}$$

4. Consider the rotation matrix obtained by rotating an initial set of axes counterclockwise by ϕ about the z-axis, then rotated about the new x-axis counterclockwise by θ, and finally rotated about the new z-axis counterclockwise by ψ. These are the **Euler angles** and the corresponding rotation matrix is

$$\begin{pmatrix} \cos\psi\cos\phi - \cos\theta\sin\phi\sin\psi & \cos\psi\sin\phi + \cos\theta\cos\phi\sin\psi & \sin\psi\sin\theta \\ -\sin\psi\cos\phi - \cos\theta\sin\phi\cos\psi & -\sin\psi\sin\phi + \cos\theta\cos\phi\cos\psi & \cos\psi\sin\theta \\ \sin\theta\sin\phi & -\sin\theta\cos\phi & \cos\theta \end{pmatrix}$$

Verify that the angle of rotation φ of this transformation is given by

$$\cos\left(\tfrac{1}{2}\varphi\right) = \cos\left[\tfrac{1}{2}(\phi+\psi)\right]\cos\left(\tfrac{1}{2}\theta\right)$$

5. Determine the axis of the transformation in Problem 4.

6.* Verify that the direct product of two irreducible representations of SO(3) has the following decomposition

$$\chi^{(\ell_1)}(\varphi)\chi^{(\ell_2)}(\varphi) = \sum_{\ell=|\ell_1-\ell_2|}^{\ell_1+\ell_2} \chi^{(\ell)}(\varphi)$$

This is called the **Clebsch–Gordan series** and provides a group-theoretic statement of the addition of angular momenta.

7.* Determine the corresponding Clebsch-Gordan series for SO(2).

8.* Show that the requirement that $xx^* + yy^*$ is invariant under the complex transformation

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

together with the determinant of this transformation being unity means that the transformation must have the form

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

where $aa^* + bb^* = 1$.

Group Theory

Solutions to Problem Set 10 December 14, 2001

1. As shown in Section 8.3.1, the eigenvalues of an orthogonal matrix have modulus unity. These eigenvalues are also the roots of the polynomial equation $\det(A - \lambda I) = 0$, so the Fundamental Theorem of Algebra requires that, if these roots are complex, they must occur in complex conjugate pairs. Thus, only in an *odd-dimensional* space is there guaranteed to be a single real eigenvalue of unity. The corresponding eigenvector is the axis of rotation.

2. If the fixed point is taken as the origin of the set of axes of the body, then the displacement of the rigid body involves no translation, but only a change of orientation, i.e., a rotation. Since, in three dimensions, every rotation can be expressed in an axis-angle representation, Euler's theorem follows immediately.

3. The general improper transformation in two dimensions is

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\varphi & \sin\varphi \\ \sin\varphi & -\cos\varphi \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} .$$

Thus, for the functions $(x \pm iy)^m$ we have

$$\begin{aligned} (x' \pm iy')^m &= \Big[x\cos\varphi + y\sin\varphi \pm i(x\sin\varphi - y\cos\varphi)\Big]^m \\ &= \Big[x(\cos\varphi \pm i\sin\varphi) \mp iy(\cos\varphi \pm i\sin\varphi)\Big]^m \\ &= (x \mp iy)^m\, \mathrm{e}^{\pm im\varphi} , \end{aligned}$$

so they generate the representation

$$\begin{bmatrix} (x' + iy')^m \\ (x' - iy')^m \end{bmatrix} = \begin{pmatrix} 0 & \mathrm{e}^{im\varphi} \\ \mathrm{e}^{-im\varphi} & 0 \end{pmatrix} \begin{bmatrix} (x + iy)^m \\ (x - iy)^m \end{bmatrix} .$$

To determine whether or not this representation is reducible, we apply Schur's first lemma. Suppose a matrix A commutes with all of the matrices of our two dimensional representation. Then, we have

$$\underbrace{\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} 0 & e^{im\varphi} \\ e^{-im\varphi} & 0 \end{pmatrix}}_{\begin{pmatrix} a_{12}e^{-im\varphi} & a_{11}e^{im\varphi} \\ a_{22}e^{-im\varphi} & a_{21}e^{im\varphi} \end{pmatrix}} = \underbrace{\begin{pmatrix} 0 & e^{im\varphi} \\ e^{-im\varphi} & 0 \end{pmatrix}\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}}_{\begin{pmatrix} a_{21}e^{im\varphi} & a_{22}e^{im\varphi} \\ a_{11}e^{-im\varphi} & a_{12}e^{-im\varphi} \end{pmatrix}} .$$

Thus, if $m \neq 0$, we must require that $a_{12} = a_{21} = 0$ and that $a_{11} = a_{22}$, i.e., A is multiple of the 2×2 unit matrix and, according to Schur's first lemma, this representation is *irreducible.* However, of $m = 0$, we need only require that $a_{12} = a_{21}$ and $a_{11} = a_{22}$, so this is a *reducible* representation.

4. The rotation angle φ is calculated from the trace of the transformation matrix:

$$\begin{aligned} 1 + 2\cos\varphi &= \cos\psi\cos\phi - \cos\theta\sin\phi\sin\psi - \sin\psi\sin\phi \\ &\qquad + \cos\theta\cos\phi\cos\psi + \cos\theta \\ &= (1+\cos\theta)(\cos\phi\cos\psi - \sin\phi\sin\psi) + \cos\theta \\ &= (1+\cos\theta)\cos(\phi+\psi) + \cos\theta\,. \end{aligned}$$

Using the triginometric identity

$$1 + 2\cos\varphi = 4\cos^2\left(\tfrac{1}{2}\varphi\right) - 1\,,$$

we obtain

$$\begin{aligned} 4\cos^2\left(\tfrac{1}{2}\varphi\right) &= (1+\cos\theta)[1+\cos(\phi+\psi)] \\ &= 4\cos^2\left(\tfrac{1}{2}\theta\right)\cos^2\left[\tfrac{1}{2}(\phi+\psi)\right], \end{aligned}$$

or,

$$\cos\left(\tfrac{1}{2}\varphi\right) = \cos\left(\tfrac{1}{2}\theta\right)\cos\left[\tfrac{1}{2}(\phi+\psi)\right].$$

5. The axis of the transformation in Problem 4 is determined from the equations derived in Section 8.3.2:

$$\frac{n_2}{n_1} = \frac{a_{31}-a_{13}}{a_{23}-a_{32}}, \qquad \frac{n_3}{n_1} = \frac{a_{12}-a_{21}}{a_{23}-a_{32}}.$$

The denominator of these expressions is

$$a_{23}-a_{32} = \sin\theta\cos\psi + \sin\theta\cos\phi = \sin\theta(\cos\psi+\cos\phi).$$

We also have

$$\begin{aligned} a_{31}-a_{13} &= \sin\theta\sin\phi - \sin\theta\sin\psi = \sin\theta(\sin\phi-\sin\psi) \\ a_{12}-a_{21} &= \cos\psi\sin\phi + \cos\theta\cos\phi\sin\psi \\ &\qquad + \sin\psi\cos\phi + \cos\theta\sin\phi\cos\psi \\ &= (1+\cos\theta)(\cos\phi\sin\psi + \sin\phi\cos\psi) \\ &= (1+\cos\theta)\sin(\phi+\psi). \end{aligned}$$

Thus, the (unnormalized) direction of the rotation axis is

$$\left\{1,\ \frac{\sin\phi-\sin\psi}{\cos\psi+\cos\phi},\ 2\frac{(1+\cos\theta)\sin(\phi+\psi)}{\sin\theta(\cos\phi+\cos\psi)}\right\}.$$

6. There are a number of ways of decomposing the direct product of irreducible representations of SO(3). The books by Tinkham

and Jones give two very different approaches. Below, we provide a third method. We first calculate the direct product

$$\chi^{(\ell)}(\varphi)\chi^{(1)}(\varphi) = \left(\sum_{m=-\ell}^{\ell} \mathrm{e}^{-im\varphi} \right) \left(\sum_{m_1=-1}^{1} \mathrm{e}^{-im_1\varphi} \right).$$

By expanding the second summation and multiplying the first summation with each of the exponentials, we obtain

$$\left(\sum_{m=-\ell}^{\ell} \mathrm{e}^{-im\varphi} \right) \left(\sum_{m_1=-1}^{1} \mathrm{e}^{-im_1\varphi} \right) = \sum_{m=-\ell}^{\ell} \mathrm{e}^{-im\varphi} \left(\mathrm{e}^{i\varphi} + 1 + \mathrm{e}^{-i\varphi} \right)$$

$$= \sum_{m=-\ell}^{\ell} \mathrm{e}^{-i(m-1)\varphi} + \sum_{m=-\ell}^{\ell} \mathrm{e}^{-im\varphi} + \sum_{m=-\ell}^{\ell} \mathrm{e}^{-i(m+1)\varphi}.$$

If, in the first summation on the right-hand side of this equation, we change the summation variable to $m' = m - 1$ and in the last summation change the summation variable to $m' = m + 1$, we have

$$\begin{aligned} \sum_{m=-\ell}^{\ell} \mathrm{e}^{-i(m-1)\varphi} &+ \sum_{m=-\ell}^{\ell} \mathrm{e}^{-i(m+1)\varphi} \\ &= \sum_{m'=-\ell-1}^{\ell-1} \mathrm{e}^{-im'\varphi} + \sum_{m'=-\ell+1}^{\ell+1} \mathrm{e}^{-im'\varphi} \\ &= \sum_{m'=-(\ell+1)}^{\ell+1} \mathrm{e}^{-im'\varphi} + \sum_{m'=-(\ell-1)}^{\ell-1} \mathrm{e}^{-im'\varphi}. \end{aligned}$$

In fact, for any positive integer k, we have

$$\begin{aligned} \sum_{m=-\ell}^{\ell} \mathrm{e}^{-i(m-k)\varphi} &+ \sum_{m=-\ell}^{\ell} \mathrm{e}^{-i(m+k)\varphi} \\ &= \sum_{m'=-\ell-k}^{\ell-k} \mathrm{e}^{-im'\varphi} + \sum_{m'=-\ell+k}^{\ell+k} \mathrm{e}^{-im'\varphi} \\ &= \sum_{m'=-(\ell+k)}^{\ell+k} \mathrm{e}^{-im'\varphi} + \sum_{m'=-(\ell-k)}^{\ell-k} \mathrm{e}^{-im'\varphi}. \qquad (1) \end{aligned}$$

Thus, we conclude that

$$\chi^{(\ell)}(\varphi)\chi^{(1)}(\varphi) = \chi^{(\ell-1)}(\varphi) + \chi^{(\ell)}(\varphi) + \chi^{(\ell+1)}(\varphi)\,.$$

Then, by using (1), we have, in the general case

$$\begin{aligned}\chi^{(\ell)}(\varphi)\chi^{(\ell')}(\varphi) &= \sum_{m=-\ell}^{\ell} \mathrm{e}^{-im\varphi}\Big[\mathrm{e}^{i\ell'\varphi} + \mathrm{e}^{i(\ell'-1)\varphi} + \cdots + \mathrm{e}^{-i\ell'\varphi}\Big] \\ &= \chi^{(\ell+\ell')}(\varphi) + \chi^{(\ell+\ell'-1)}(\varphi) + \cdots + \chi^{(\ell-\ell')}\,.\end{aligned}$$

Therefore,

$$\chi^{(\ell)}(\varphi)\chi^{\ell'}(\varphi) = \sum_{m=\ell-\ell'}^{\ell+\ell'} \chi^{(m)}(\varphi)\,,$$

where, from our procedure, it is clear that $\ell' \leq \ell$.

7. The corresponding Clebsch–Gordan series for SO(2) is very simple because the group is Abelian. Since

$$\chi^{(m)}(\varphi) = \mathrm{e}^{im\varphi}\,,$$

then

$$\chi^{(m_1)}(\varphi)\chi^{(m_2)}(\varphi) = \chi^{(m_1+m_2)}(\varphi)\,.$$

8. Given the complex transformation

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}, \qquad (2)$$

then the invariance of the quantity $xx^* + yy^*$ yields

$$\begin{aligned} & x'x'^* + y'y'^* \\ & = (ax + by)(a^*x^* + b^*y^*) + (cx + dy)(c^*x^* + d^*y^*) \\ & = (aa^* + cc^*)xx^* + (ab^* + cd^*)xy^* + (a^*b + c^*d)x^*y \\ & + (cc^* + dd^*)yy^* \, . \end{aligned}$$

Maintaining equality for all independent variations of x and y requires that

$$aa^* + cc^* = 1, \qquad ab^* + cd^* = 0, \qquad cc^* + dd^* = 1 \, . \tag{3}$$

A fourth condition is that the determinant of the transformation in (2) is unity:

$$ad - bc = 1 \tag{4}$$

If we take the second of equations (3), multiply by a^*, and then use the first of these equations and Equation (4), we obtain

$$\begin{aligned} a^*(ab^* + cd^*) &= aa^*b^* + a^*cd^* \\ &= (1 - cc^*)b^* + (1 + b^*c^*)c \\ &= b^* + c = 0 \end{aligned}$$

which yields

$$c = -b^*$$

The second of equations (4) then immediately yields

$$a = d^* \tag{5}$$

Thus, the transformation (2) must have the form

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

where $aa^* + bb^* = 1$.